**MathsHUBS**
Resources

**National Centre**
for Excellence in the
Teaching of Mathematics

**Mastery Professional Development**
*2 Operating on number*

PDF

2.2 Solving linear equations

Guidance document | Key Stage 3

## Making connections

The NCETM has identified a set of six 'mathematical themes' within Key Stage 3 mathematics that bring together a group of 'core concepts'.

The second of these themes is *Operating on number*, which covers the following interconnected core concepts:

2.1     Arithmetic procedures

2.2     **Solving linear equations**

This guidance document breaks down core concept *2.2 Solving linear equations* into four statements of knowledge, skills and understanding:

2.2.1   Understand what is meant by finding a solution to a linear equation with one unknown

2.2.2   Solve a linear equation with a single unknown on one side where obtaining the solution requires one step

2.2.3   Solve a linear equation with a single unknown where obtaining the solution requires two or more steps (no brackets)

2.2.4   Solve efficiently a linear equation with a single unknown involving brackets

Then, for each of these statements of knowledge, skills and understanding we offer a set of key ideas to help guide teacher planning.

*Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.*

## Overview

It is important for students to appreciate that number and algebra are connected, and that the solving of equations is essentially concerned with operations on, as yet, unknown numbers. This core concept builds on students' introduction to the language of algebra at Key Stage 2. It explores how linear equations are effectively the formulation of a series of operations on unknown numbers, and how the solving of such equations is concerned with undoing these operations to find the value of the unknown.

Understanding the '=' sign as 'having the same value as', and the correct use of order of operations, along with inverse operations, are key to the solving of equations. Students also need to understand the difference between an expression and an equation, and the different roles that letters might take. For example, $3x + 7$ is an expression where the variable $x$, and therefore the expression as a whole, can take an infinite number of values. It also has a duality about it – it is a process and the result of that process. It is a way of describing a set of operations on a variable (i.e. multiply by three and add seven), as well as a way of representing the actual result when $x$ is multiplied by three and seven is added. When some restriction is put on this expression, as in $3x + 7 = 10$, the letter $x$ ceases to represent a variable but is now an unknown, the specific value of which will make the equation true. It is important that students experience this sense of the infinite (as in the values an expression can take) and the finite (specific values to satisfy an equation). The use of coordinates and graphs is very helpful in this regard as they provide a way of representing such situations to:

- reveal particular values for $x$ (inputs) giving particular values for the expression (outputs)
- get a sense of the range of different values that an expression can take
- encapsulate an infinity of values in one picture
- home in on one point where a solution is satisfied.
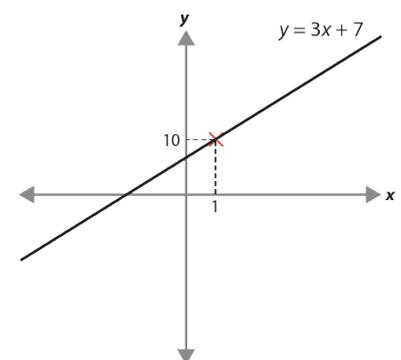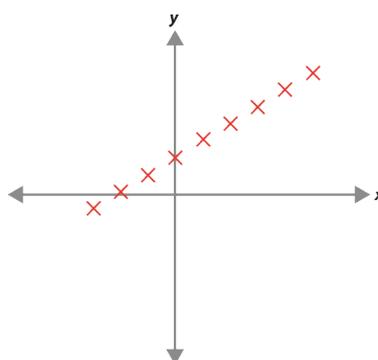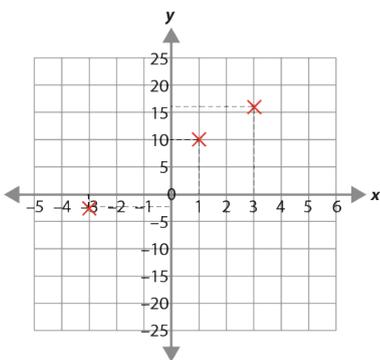


Students should also experience doing and undoing in the context of equations to develop their understanding of how to perform the correct inverse operation, in the correct order. Strategies, such as 'building up' equations by starting with a simple '$x = 3$', and developing this by operating on both sides to create increasingly complex equations, may support students with this. Students also need to be given opportunities to work on examples that lead to a range of solutions, including positive, negative and fractional.

Much of this learning is new and is built upon in Key Stage 4; therefore, it is essential that students are given time to develop a secure and deep understanding of these important ideas and techniques.

## Prior learning

Before beginning to teach *Solving linear equations* at Key Stage 3, students should already have a secure understanding of the following from previous study:

| Key stage | Learning outcome |
|---|---|
| Upper Key Stage 2 | • Express missing number problems algebraically<br>• Find pairs of numbers that satisfy an equation with two unknowns<br>• Enumerate possibilities of combinations of two variables |
| Key Stage 3 | • 1.4.1 Understand and use the conventions and vocabulary of algebra including forming and interpreting algebraic expressions and equations<br>• 1.4.2 Simplify algebraic expressions by collecting like terms to maintain equivalence<br>• 1.4.3 Manipulate algebraic expressions using the distributive law to maintain equivalence<br>• 2.1.1 Understand and use the structures that underpin addition and subtraction strategies<br>• 2.1.2 Understand and use the structures that underpin multiplication and division strategies<br>• 2.1.3 Know, understand and use fluently a range of calculation strategies for addition and subtraction of fractions<br>• 2.1.4 Know, understand and use fluently a range of calculation strategies for multiplication and division of fractions<br>• 2.1.5 Use the laws and conventions of arithmetic to calculate efficiently<br><br>**Please note:** Numerical codes refer to statements of knowledge, skills and understanding in the NCETM breakdown of Key Stage 3 mathematics. |

You may find it useful to speak to your partner schools to see how the above has been covered and the language used.

You can find further details regarding prior learning in the following segments of the NCETM primary mastery professional development materials[1]:

- Year 4: 2.10 Connecting multiplication and division, and the distributive law
- Year 5: 1.28 Common structures and the part–part–whole relationship
- Year 5: 1.29 Using equivalence and the compensation property to calculate
- Year 5: 2.18 Using equivalence to calculate
- Year 5: 2.22 Combining multiplication with addition and subtraction
- Year 6: 1.31 Problems with two unknowns
- Year 6: 2.28 Combining division with addition and subtraction

## Checking prior learning

The following activities from the [NCETM primary assessment materials](NCETM primary assessment materials)[2] offer useful ideas for assessment, which you can use in your classes to check whether prior learning is secure:

| Reference | Activity |
|---|---|
| Year 6 page 29 | *Which of the following statements do you agree with?*<br>*Explain your decisions.*<br><br>• *The value 5 satisfies the symbol sentence $3 \times \boxed{\phantom{x}} + 2 = 17$*<br><br>• *The value 7 satisfies the symbol sentence $3 + \boxed{\phantom{x}} \times 2 = 10 + \boxed{\phantom{x}}$*<br><br>• *The value 6 solves the equation $20 - x = 10$*<br><br>• *The value 5 solves the equation $20 \div x = x - 1$* |
| Year 6 page 29 | *I am going to buy some 10 p stamps and some 11 p stamps.*<br>*I want to spend exactly 93 p.*<br><br>*Write this as a symbol sentence and find whole number values that satisfy your sentence.*<br><br>*Now tell me how many of each stamp I should buy.* |

## Key vocabulary

| Term | Definition |
|---|---|
| coefficient | Often used for the numerical coefficient. More generally, a factor of an algebraic term.<br>Example 1: In the term $4xy$, 4 is the numerical coefficient of $xy$ but $x$ is also the coefficient of $4y$ and $y$ is the coefficient of $4x$.<br>Example 2: in the quadratic equation $3x^2 + 4x - 2$, the coefficients of $x^2$ and $x$ are 3 and 4 respectively. |
| equation | A mathematical statement showing that two expressions are equal. The expressions are linked with the symbol $=$<br>Examples: $7 - 2 = 4 + 1$    $4x = 3$    $x^2 - 2x + 1 = 0$ |
| linear | In algebra, describing an expression or equation of degree one.<br>Example: $2x + 3y = 7$ is a linear equation.<br>All linear equations can be represented as straight line graphs. |
| solution | A solution to an equation is a value of the variable that satisfies the equation, i.e. when substituted into the equation, makes it true. |

| | |
|---|---|
| | A solution set is the set of values that satisfy a given set of equations or inequalities. |
| unknown | A number that is not known.<br><br>Example: In the expression $2x - 5$, $x$ represents an unknown.<br><br>When presented with more information, such as in the form of an equation (e.g. $2x - 5 = 6$), this unknown can be found. |
| variable | A quantity that can take on a range of values, often denoted by a letter, $x, y, z, t, \ldots$, etc. |

## Collaborative planning

Below we break down each of the four statements within *Solving linear equations* into a set of key ideas to support more detailed discussion and planning within your department. You may choose to break them down differently depending on the needs of your students and timetabling; however, we hope that our suggestions help you and your colleagues to focus your teaching on the key points and avoid conflating too many ideas.

**Please note:** We make no suggestion that each key idea represents a lesson. Rather, the 'fine-grained' distinctions we offer are intended to help you think about the learning journey irrespective of the number of lessons taught. Not all key ideas are equal in length and the amount of classroom time required for them to be mastered will vary, but each is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

The following letters draw attention to particular features:

**D** Suggested opportunities for **deepening** students' understanding through encouraging mathematical thinking.

**L** Examples of shared use of **language** that can help students to understand the structure of the mathematics. For example, sentences that all students might say together and be encouraged to use individually in their talk and their thinking to support their understanding (for example, *'The smaller the denominator, the bigger the fraction.'*).

**R** Suggestions for use of **representations** that support students in developing conceptual understanding as well as procedural fluency.

**V** Examples of the use of **variation** to draw students' attention to the important points and help them to see the mathematical structures and relationships.

**PD** Suggestions of questions and prompts that you can use to support a **professional development** session.

For selected key ideas, marked with an asterisk (*), we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches, together with suggestions and prompts to support professional development and collaborative planning. You can find these at the end of the set of key ideas.

## Key ideas

### 2.2.1  Understand what is meant by finding a solution to a linear equation with one unknown

In Key Stage 2, students were introduced to the use of symbols and letters to represent variables and unknowns in mathematical situations. Therefore, students should be able to express missing-number problems algebraically and find pairs of numbers that satisfy an equation with two unknowns. Key Stage 3 builds on this experience by providing opportunities for students to understand the concept of a 'solution' to a (linear) equation.

A useful way of supporting students' appreciation of what makes a solution is to offer them the opportunity to create their own equations starting from a given value, as in this 'spider diagram':

$3(x + 2) - 1 = 20$

$3(x + 2) = 21$

$x + 2 = 7$

$2(\frac{x}{5} + 3) = 8$

$\frac{x}{5} + 3 = 4$

$\frac{x}{5} = 1$

$x = 5$

$2x = 10$

$x - 7 = -2$

$2x + 5 = 15$

$3(x - 7) = -6$

$\frac{2x + 5}{3} = 5$

$3(x - 7) + 10 = 4$

$\frac{3(x - 7) + 10}{2} = 2$
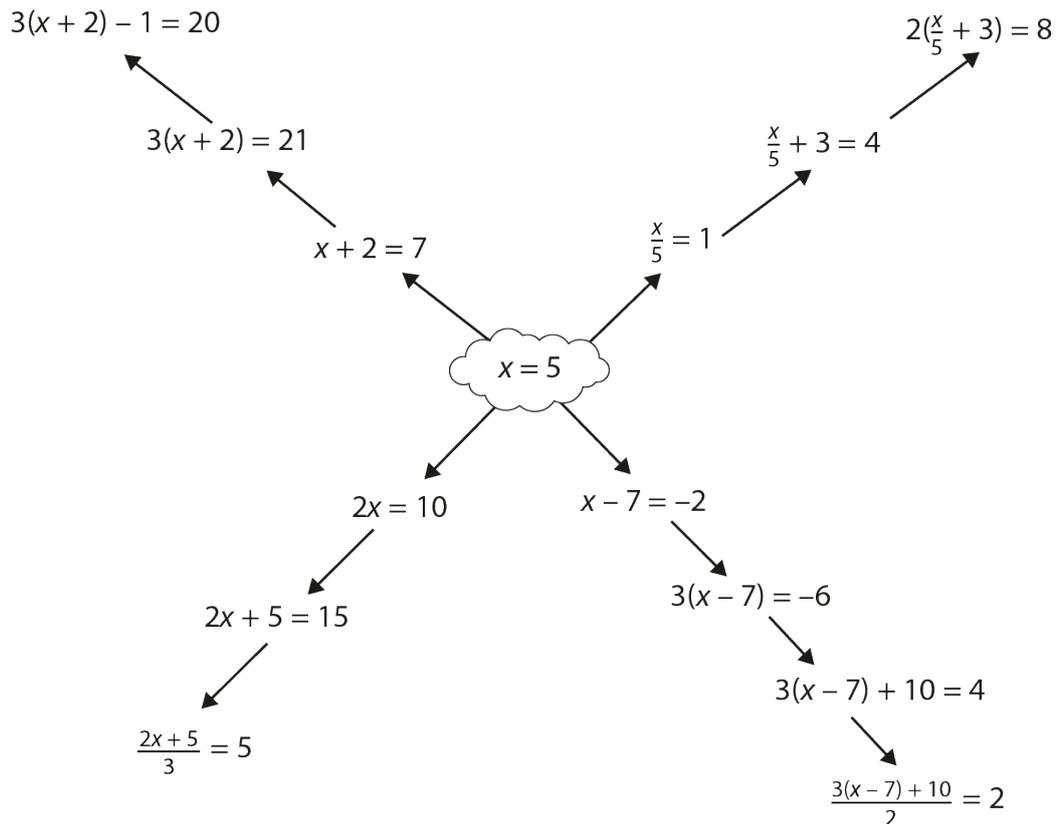
Students should appreciate that $x = 5$ is a linear equation (to which the solution is obvious) and that all other linear equations which are a transformation of this have the same solution. This also links to the awareness that linear equations have only one solution.

It is important that students do not just learn and blindly follow a set of procedural rules for solving equations without this sense of what a solution means. Deep, conceptual understanding allows students to be fluent and flexible problem-solvers.

In addition to finding solutions to equations, it is also helpful to present students with a range of examples and non-examples, including:

- reasoning whether certain values are or are not solutions to particular equations. For example, *'Samira says that x = 3 is a solution to the equation 7 − 5x = 8. Is she right? If so, explain why; if not, explain why not and correct her.'*
- interrogating equations that do not have a solution and explaining why. For example, *'What happens when you try to solve 4x + 6 = 4x? What does this mean? Why is there no solution?'*

2.2.1.1   Recognise that there are many different types of equations of which linear is one type

2.2.1.2   Understand that in an equation the two sides of the 'equals' sign balance

2.2.1.3*  Understand that a solution is a value that makes the two sides of an equation balance

2.2.1.4   Understand that a family of linear equations can all have the same solution

## 2.2.2   Solve a linear equation with a single unknown on one side where obtaining the solution requires one step

Building on Key Stage 2 experiences, this collection of key ideas explores how simple, one-step linear equations are the formulation of one operation on an unknown number, and how these equations can be solved by undoing the operation to find the value of the unknown. If students have previously mastered additive and multiplicative structures, they should be able to recognise alternative versions of the family of four within that structure, e.g. $5 + 3 = 8$, so $8 − 3 = 5$, $8 − 5 = 3$ and $3 + 5 = 8$. This understanding will enable them to construct all four rearrangements of the equation $x + 3 = 10$ ($3 + x = 10$, $10 − 3 = x$, $10 − x = 3$).

A similar process can be followed for equations of the form $x − a = b$, $ax = b$ and $\frac{x}{a} = b$. In each case, one of the rearrangements will result in a form from which the solution can be calculated.

An important awareness is that, if $a = b$, then $a + c = b + c$ and $a \times c = b \times c$. These ideas could usefully form the basis of separate lessons.

2.2.2.1   Solve a linear equation requiring a single additive step

2.2.2.2   Solve a linear equation requiring a single multiplicative step

## 2.2.3   Solve a linear equation with a single unknown where obtaining the solution requires two or more steps (no brackets)

Building on solving simple linear equations requiring one step, this section explores how linear equations can also be the formulation of more than one operation on unknown numbers, and how the solving of these equations is concerned with the undoing of these operations in the correct order to find the value of the unknown.

When using the balance method (i.e. operating in the same way on both sides of an equation to maintain equality), it will be useful to explore what constitutes the most efficient solution with students. For example, in the equation $5x − 14 = 6$, it is important for students to understand that any operation applied to both sides of the equation will result in equality being maintained and to reason why some operations lead to a solution more quickly than others.

For example, students who choose to divide both sides by five, giving $\frac{5x − 14}{5} = \frac{6}{5}$, should be encouraged to do this, to compare it with other possible operations and to reason why (although not incorrect) this is not a wise decision. A revisiting of the type of 'spider diagram' that was introduced in 2.2.1 (above) might support this reasoning when considering what order of operations was followed when tracking from the outside of the diagram back to the central solution.

Students will benefit from exploring these ideas with a wide variety of linear equations with unknowns on both sides and, through these experiences, become aware that all equations of the type $ax + b = cx + d$ can be reduced to the form $Ax + B = C$.

Exploring equations such as $6 − 2x = x + 9$ will usefully give rise to a discussion about whether to subtract $x$ from both sides or to add $2x$ to both sides. Such discussions of efficiency and ease of

calculation will support the development of approaches to solving equations of the form $a - x = b$, which typically students find difficult.

Similarly, consideration of equations of the form $\frac{a}{x} = b$ and $\frac{a}{x} + c = b$ will help students see that these can be transformed into $a = bx$ and $a + cx = bx$.

2.2.3.1   Understand that an equation needs to be in a format to be 'ready' to be solved, through collecting like terms on each side of the equation

2.2.3.2   Know that when an additive step and a multiplicative step are required, the order of operations will not affect the solution

2.2.3.3*  Recognise that equations with unknowns on both sides of the equation can be manipulated so that the unknowns are on one side

2.2.3.4   Solve complex linear equations, including those involving reciprocals

## 2.2.4   Solve efficiently a linear equation with a single unknown involving brackets

By considering a range of linear equations involving brackets, students should explore the importance of noticing the structure of an equation in order to decide on the most efficient method for solving it. For example, $3(x - 2) = 27$ can be simplified directly to $x - 2 = 9$ rather than multiplying out the brackets first.

Through discussion, students can secure and deepen their understanding of solving linear equations, and reflect on the efficiency and elegance of the solutions. Using a range of examples that prompt discussions about method choices will be important. For instance, the following examples could be used to highlight when it is useful to use common factors to simplify, rather than multiplying out the brackets:

$$2(x + 1) + 3(x + 2) = 10$$
$$2(x + 1) + 3(x + 1) = 10$$
$$2(x + 1) + 2(x + 2) = 10$$

Attention may also be given to the way in which different representations of the same equation may suggest different methods. For example, in the equation $\frac{1}{3}(x + 3) = 5$, students may want to expand the brackets, while the same equation represented as $\frac{x + 3}{3} = 5$ may lead students to multiplying by three as a first step. Students should be made aware of these different representations in order to make informed and flexible decisions about the most efficient route to a solution.

2.2.4.1   Appreciate the significance of the bracket in an equation

2.2.4.2   Recognise that there is more than one way to remove a bracket when solving an equation

2.2.4.3   Solve equations involving brackets where simplification is necessary first

## Exemplified key ideas

### 2.2.1.3    Understand that a solution is a value that makes the two sides of an equation balance

**Common difficulties and misconceptions**

The shift from understanding a letter symbol as a variable (*'x can be any number'*), to solving an equation (*'What is the value of x?'*), can be a challenge for students who do not understand that the solution to an equation is a snapshot of the expression at one point as the variable changes.

Students might view an equation such as $4x + 3 = 7$ as an invitation to start a process, subtracting three and dividing by four, without necessarily understanding what the process is leading to (other than the answer to the question). It is important that students understand what it means to solve an equation – that the expressions that form the equation now share the same value. The solution to the equation identifies the value of $x$ at which that equality can be found. Therefore, students should understand that if they have found a solution to the equation, they can easily check its accuracy themselves, by substituting it back into the equation. This can be very empowering.

As students progress to solving more complex equations, it is important that they have a deep understanding of the meaning of algebraic expressions and of equality. When solving one- or two-stage equations (such as $4x + 3 = 7$), a common approach is to think of the expression $4x + 3$ as describing a sequence of operations on $x$ (i.e. multiply by four and then add three). Because the result of this sequence of operations on $x$ is seven, the solution process can be thought of as operating on seven by reversing this sequence (i.e. subtracting three and then dividing by four), as shown in this function diagram:
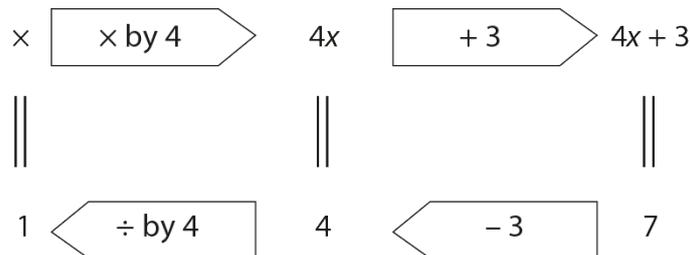


However, such an approach does not work with more complex linear equations, such as $4x + 3 = 3x + 10$, so it is important that students have an alternative way of thinking about expressions and equality.

This requires students to see an expression, such as $4x + 3$, not as a sequence of operations, but as the result of such a sequence. In this metaphor, both sides of the equation have the same value, and as long as any transformation that is applied is applied to both sides, equality will be maintained. This sense of an equation is captured in the classic balance diagram:

| $4x + 3$ | | $7$ |
|---|---|---|

$-3$ ↓     ↓ $-3$

| $4x$ | | $4$ |
|---|---|---|

$\div 4$ ↓     ↓ $\div 4$

| $x$ | | $1$ |
|---|---|---|
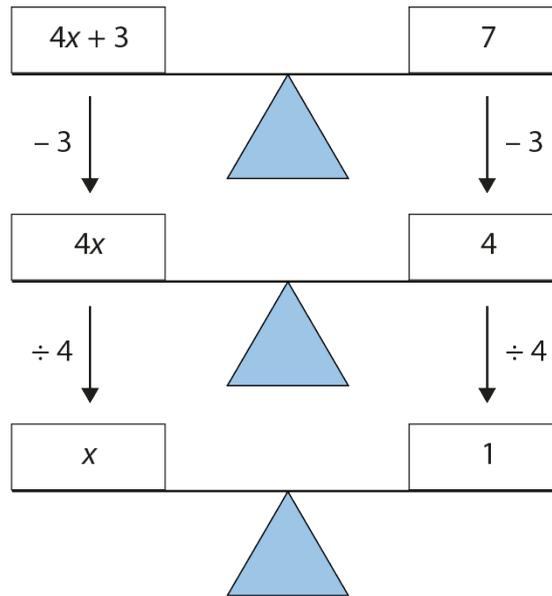
This is not an 'either/or' situation, and students will benefit from having both senses of an expression and an equation, and being able to understand both methods of solution. In fact, they complement each other. The notion of an expression as a sequence of operations (as in the first 'doing and undoing' approach) helps students see which transformations to apply to both sides of the equation, and in what order, when using the second 'balance' approach.

| What students need to understand | Guidance, discussion points and prompts |
|---|---|
| Understand that the solution to an equation is a particular snapshot of a relationship between a variable and an expression. <br><br> *Example 1:* <br><br> *What's the same and what's different about these three equations?* <br><br> *A: m + n = 10* <br><br> *B: 7 + n = 10* <br><br> *C: m + m = 10* | **V**   The intention in *Example 1* is to draw students' attention to the way in which a linear equation has one fixed solution, while a formula, such as that offered in equation *A*, has a range of possible correct solutions. <br><br> Students who may have worked with letters representing only variables may find the shift to there being just one correct value for the letter symbol to be a challenging one. This question may give an opportunity to confront this challenge explicitly. |

*Example 2:*

*This table shows the outcome of substituting different values of p into the expressions 3p + 5 and 5p − 1 calculated using a spreadsheet.*

| p | 3p + 5 | 5p − 1 |
|---|---|---|
| −5 | −10 | −26 |
| −4 | −7 | −21 |
| −3 | −4 | −16 |
| −2 | −1 | −11 |
| −1 | 2 | −6 |
| 0 | 5 | −1 |
| 1 | 8 | 4 |
| 2 | 11 | 9 |
| 3 | 14 | 14 |
| 4 | 17 | 19 |
| 5 | 20 | 24 |
| 6 | 23 | 29 |
| 7 | 26 | 34 |
| 8 | 29 | 39 |
| 9 | 32 | 44 |
| 10 | 35 | 49 |
| 11 | 38 | 54 |
| 12 | 41 | 59 |

*Use the table to write down:*

a)  *The value of 3p + 5 when p = 7.*

b)  *The value of 5p − 1 when p = 7.*

c)  *The value of p when 5p − 1 = 29.*

d)  *The value of p when 3p + 5 = 29.*

**R** The role of the letter as a variable is made explicit in this representation, as two different expressions are evaluated with increasing values of $p$.

In parts a) and b), students have the opportunity to see that the same value of $p$ results in each expression taking a different value.

In parts c) and d), students' attention should be drawn to the fact that, when given the outcome of evaluating an expression, they are able to identify the value of $p$, and that the same outcome has resulted from different values of $p$. Students should also be aware that (for these linear expressions) there is a unique value of $p$ for each outcome of each expression.

*Example 3:*

*This line graph shows the value of 3p + 5 for different values of p.*
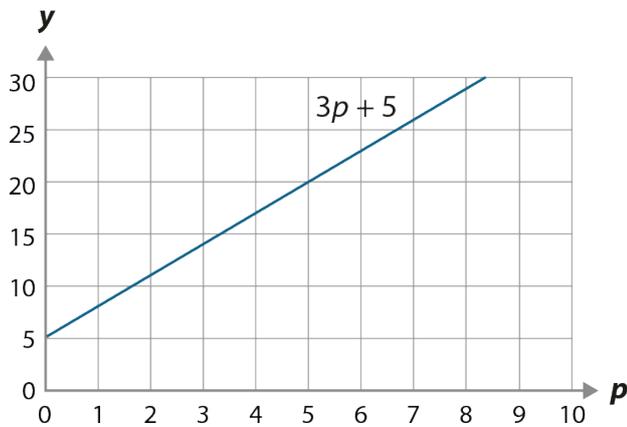


a) *Use the line graph to write down the value of 3p + 5 when p = 7.*

*This line graph shows the value of 5p − 1 for different values of p.*
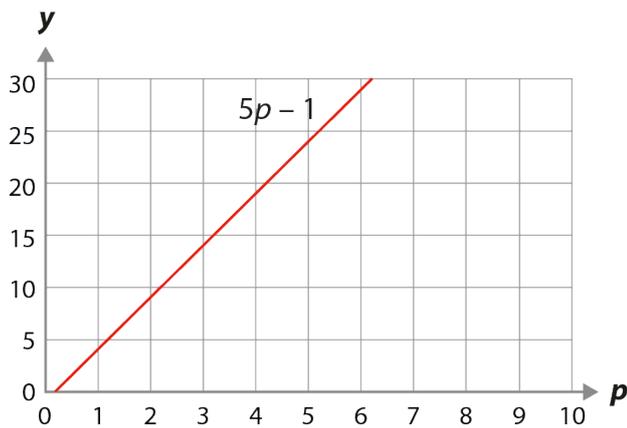


b) *Use this line graph to write down the value of 5p − 1 when p = 7.*

c) *Use the first line graph to write down the value of p when 3p + 5 = 29.*

d) *Use the second line graph to write down the value of p when 5p − 1 = 29.*

**R**  In *Example 3*, the same expressions and prompts are used as in *Example 2*, but the representation has been changed from a table of values to a graph. The key awareness for students here is that *p* represents a variable, but at certain points a snapshot of that variable can be taken and statements made about the values. For example, when $p = 7$, then $3p + 5 = 26$ and when $3p + 5 = 29$, then $p = 8$. The graphs may represent the continuous nature of the variable more accurately but be less intuitive for students to interpret, so it is important to give time for them to understand the connection between the graphs and the table from *Example 2*.

Understand that the solution to an equation is the value of the variable at which two expressions are balanced.

*Example 4:*

a) *Use the table in Example 2 to find the point where the expression $3p + 5$ and the expression $5p − 1$ both share the same value.*

b) *Use the table to write down the value of $p$ when $3p + 5 = 5p − 1$.*

*Both lines have been drawn on the same axes.*
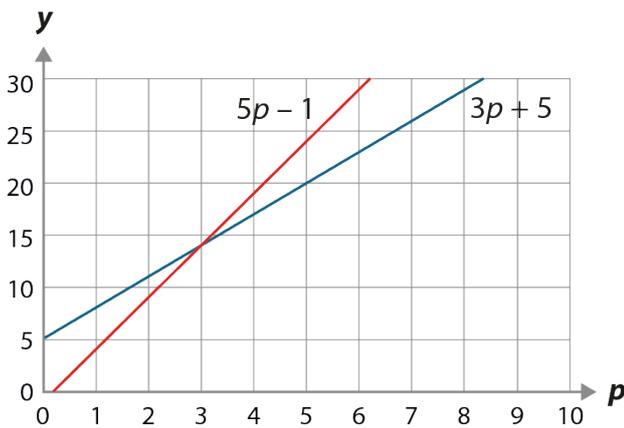


c) *Use the graph to find the point where the expression $3p + 5$ and the expression $5p − 1$ both share the same value.*

d) *Use the graph to write down the value of $p$ when $3p + 5 = 5p − 1$.*

**R** *Example 4* builds on *Example 2* and should be seen as a continuation of it. Parts a) and b) are both focusing attention on a key point, the solution to $3p + 5 = 5p − 1$, but offering slightly different interpretations. The intention is to give students the opportunity to make sense of the solution to an equation as the point at which both 'sides' of the equation have the same value.

The graph with both expressions represented offers the same connection to *Example 3*. Again, students' attention should be drawn to the point at which the two expressions share the same value, this time understanding that this is a unique point on the graph.

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*Example 5:*

*This rectangle is not drawn to scale.*



**R** In *Example 5*, the representation is used to support students in developing their understanding of the letter symbol as a variable, and the continuous nature of that variable, as they imagine the ways in which the shape will change as $m$ changes.

Once students have spent time with the task, you might like to prompt them by asking them to sketch the rectangle when $m$ is *very* small, then when $m$ is *very* large, and discuss the differences in the shapes.

Part b) then fixes the relationship between the height and the width of the rectangle. Here, the representation can be used to draw attention to the equality of the expression

a) If drawn accurately, would the rectangle be short and wide, or tall and thin? Explain how you know.

b) Is there a value (or values) of m for which the shape will become a square? Explain how you know.

giving the height and the expression giving the width.

You might like to generate a set of results using a spreadsheet and a graphical representation (as in *Example 4*) to make explicit the connections between these different representations to students. Using these different representations should convince students that there will only be one point at which the height and width are equal, and that is the solution to the equation.

**DV** Some students might work on finding values in which there are different relationships between the height and width of the rectangle. For example, you could consider asking students to:

- find a value of *m* where the height is double the width
- find a value of *m* where the height is three times the width
- find a value of *m* where the width is one unit longer than the height
- find a value of *m* where the height is one unit longer than the width
- find a value of *m* where the height is five units longer than the width.

These questions give a chance to further consider the role of the equals sign in the equation and to explore the limitations of numerical results when finding solutions that are not possible in the context of the problem (such as the solution to the final prompt here).

**PD** A similar task was used in the ICCAMS[3] project, in which students were asked: *'Which is bigger, 2n or n + 2?'* It was reported[†] that 48% of lower secondary students responded 2*n*, with the reason that *'multiplication makes things bigger'*, and that only 1% gave the correct response with a good reason.

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[†] J. Hodgen, M. Brown, R. Coe & D. Küchemann, 2012, *Surveying lower secondary students' understandings of algebra and multiplicative reasoning: to what extent do particular errors and incorrect strategies indicate more sophisticated understandings?*, Seoul, 12th International Congress on Mathematical Education

What does the use of the representation add to this when focusing on the 'balance point'? Which representation (the rectangle, graph, spreadsheet, algebraic equation, or one of your own) do you feel gives the best sense of equality and balance?

*Example 6:*

*The bars in this diagram represent the expressions $3x + 11$ and $5x + 1$:*

11

1

*The value of x (the length of the first bar) can vary, and in this diagram, it is 1:*

11

1
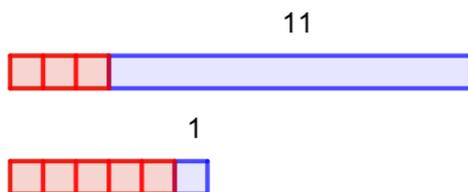
*Here are some snapshots of the bars at different values:*

*At 2.8:*

11

1

*At 4.7:*

11

1
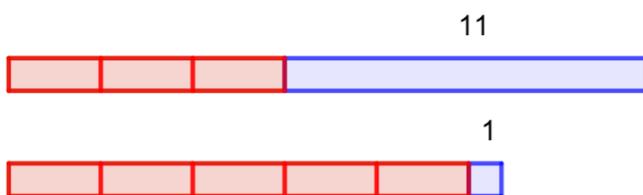
**R** Here, a bar diagram is used to show the variable nature of the letter symbol in two expressions, and to identify the point and value of *x* at which they are equal. This is an alternative way to consider the shift from the variable nature of the letter symbol in an expression to the unknown nature when working with an expression.

**PD** A GeoGebra file[4] for this representation is available online. What are the benefits of showing a dynamic representation like this? Do you feel that this offers a better insight into what it means to solve an equation?

*At 6.2:*



a) What length of the bar for x will make the top and the bottom bars have the same total length?

b) Write down the solution to the equation $3x + 11 = 5x + 1$.

*Images created using GeoGebra*

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*Example 7:*

a) (i) Explain how you would find out if $b = 9$ is a solution to the equation $13b – 11 = 106$.

   (ii) Is $b = 9$ a solution to the equation $13b – 11 = 106$?

b) (i) Explain how you would find out if $b = 9$ is a solution to the equation $10b + 15 = 105$.

   (ii) Is $b = 9$ a solution to the equation $10b + 15 = 105$?

c) Explain how you could use your previous answers to find out if $b = 9$ is also a solution to $13b – 11 = 10b + 15$.

**V** In parts a) and b) of *Example 7*, students are asked to explain how to check a solution. They might decide to do this by substitution, or by solving the original equation. Listening to their responses is likely to give an insight into their current understanding.

In part c), students are then asked to use the two previous solutions to decide on the solution to an equation (comprising the two previous expressions).

Students' attention should be drawn to the use of the equals sign and the value of each 'side' of the equation when $b = 9$, in order to deduce that the equation given in part c) cannot be correct when $b = 9$.

**D** You might like to ask students to consider whether the solution to the equation $13b – 11 = 10b + 15$ is greater or less than $b = 9$ and to justify their reasoning.

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*Example 8:*

*These scales are balanced.*
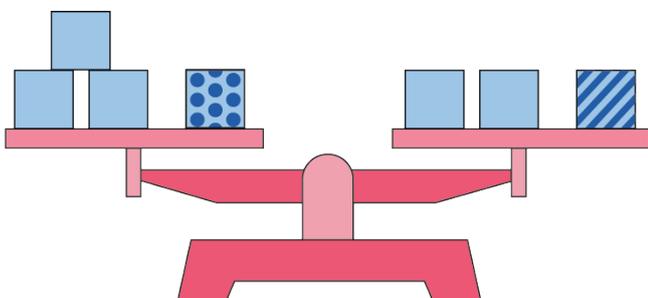


*All of the plain boxes have the same weight.*

a) Is the striped box heavier or lighter than the spotty box? Explain how you know.

**R**
**V** *Example 8* uses two different representations of the same relationship to encourage students to make explicit the connection between the balancing scales and the algebraic equation.

As in previous examples, the solution to the equation is approached by comparing the relationships present rather than by taking an algorithmic approach. The intention is to make sense of what is meant by the solution of an equation, and to see that this solution is the fixed point at which two different expressions are equal.

b) *Can you describe how much heavier or lighter the striped box is than the spotty box?*

c) *This equation describes the weight on the scales:*

$$3x + 7 = 2x + m$$

*Do you agree that m is more than 7? How much more? Explain how you know.*

d) *If we are told that m = 8, can you find a value of x that makes the scales balance? Explain how you know your answer is correct.*

When working on parts c) and d), students have more information than in the diagram, but it is presented in a way that may be less intuitive for them.

Students should be encouraged to work with both representations and to make connections between the two. They are likely to notice that the striped box has the same weight as the total of one plain box and the spotty box. They may then use the symbolic representation to identify that this means that the weight of the striped box is seven units greater than that of the spotty box.

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*Example 9:*

*Decide which of these equations has h = 5 as a solution, which has h = 0 as a solution and which has no solution.*

a) $4h + 9 = 4h + 10$

b) $7h + 5 = 3h + 5$

c) $2h = 3h - 5$

d) $5 = 5h + 5$

**V** *Example 9* offers a set of equations for students to consider. Although the question can be answered through the use of a procedure, this is not necessary. Students should be able to reason their way to a solution by using their understanding of what it means to find a solution to an equation.

The examples used here are different from those with which students might be familiar, offering zero as a solution and also no solution, and so exploring the 'edges' of what it means to be able to solve an equation.

### 2.2.3.3 Recognise that equations with unknowns on both sides of the equation can be manipulated so that the unknowns are on one side

**Common difficulties and misconceptions**

When solving an equation in which the unknown is on one side, students are able to use a 'doing and undoing' strategy to think about the equation. If students are able to interpret $3x - 2 = 10$ as *'I think of a number, multiply by three, then subtract two and the result is ten'*, then they may be able to use 'common sense' to find the original number. Function machines are commonly used to sequence the transformations made in equations and to focus on the inverse operations needed to find a solution.

When the unknown is on both sides of the equation, then 'doing and undoing' strategies break down. Students' attention has to shift to maintaining equality as the equation is transformed.

Some students may find challenging the shift in awareness needed to understand that a process (e.g. $3e$ means multiply whatever $e$ is by three) is now being manipulated as one object (e.g. in the equation $5 + 3e = 7e - 1$, we might start by subtracting $3e$). Gray and Tall (1991)[‡] coined the term 'procept' to describe symbolic notation that allows for both interpretations. Attention and thought needs to be given to supporting these students.

As students become familiar with the range of manipulations that will maintain equality, they can then transfer their focus to which of these manipulations are helpful in moving towards a solution. Here, the intention to transform the equation so that the unknown is on one side should be made explicit.

| What students need to understand | Guidance, discussion points and prompts |
|---|---|
| Understand that an equation can be considered as both a process and an object.<br><br>*Example 1:*<br><br>**c**<br>**a** \| **b**<br><br>**$3x + 7$**<br>**$3x$** \| **7**<br><br>**$8x$**<br>**$2x + 5$** \| **$6x - 5$**<br><br>a) *Look at the first bar model. Can you see:*<br>  (i)   *$a + b = c$?*<br>  (ii)  *$b + a = c$?*<br>  (iii) *$c - a = b$?*<br>  (iv) *$c - b = a$?*<br>b) *Write similar 'families', each with four equations for the other two bar models.* | **R V** In *Example 1* part a), students' awareness should be directed to the structure of the addition represented by the bar model. They should be encouraged to describe the connection between the diagram and the symbolic representation.<br><br>When working on part b), students' awareness should then be drawn to the similarity with part a), with a particular focus on the way that $3x$ or $2x + 5$ can be considered as a single object and manipulated in the same way as in the first example.<br><br>The expression $6x - 5$ in the third diagram has been chosen to raise possible common misconceptions when subtracting a negative value. Consider possible outcomes that are likely to be given by students, including $8x - 6x - 5 = 2x + 5$ or $8x - (6x - 5) = 2x + 5$, and raise the use of brackets to support the understanding of $6x - 5$ as an object. |
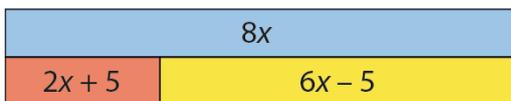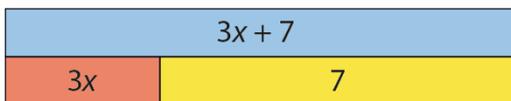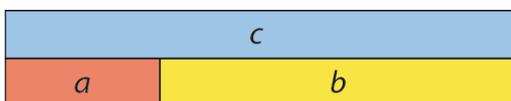
[‡] E. Gray & D. Tall, 1991, *Duality, Ambiguity and Flexibility in Successful Mathematical Thinking*, Coventry, University of Warwick

*Example 2:*

*Complete each statement.*

a) *Given that 3e = 7, then 6e = ____.*

b) *Given that 5(3w − 7) = 30, then 3w − 7 = ____.*

c) *Given that 5k = 7, then 5k + 9 = ____.*

d) *Given that 8f + 12 = 23, then 8f + 2 = ____.*

e) *Given that 3t + 6 = 2t + 1, then 3t + 10 = ____.*

**V** As in *Example 1*, students' attention in *Example 2* should be directed to the expression as an object, and the way in which this object has been transformed in the second part. The intention here is to encourage familiarity and an awareness of the way in which an expression as a whole can be transformed. This builds the understanding that there is not a need to know the value of the unknown in order to write down other valid statements. For example, a prompt for part d) might be *'What has happened to 8f + 12 to transform it into 8f + 2?'*, encouraging the idea that 8f + 12 is a single object on which a subtraction has been carried out and from which another valid equation can be written.

Understand ways in which equations can be manipulated while maintaining equality.

*Example 3:*

*Look at this spider diagram:*



| The rule for this arm is _____ each time | The rule for this arm is _____ each time |
|---|---|
| $5x + 9 = 15 + 3x$ | $2x + 12 = 18$ |
| $4x + 9 = 15 + 2x$ | $2x + 11 = 17$ |
| $3x + 9 = 15 + x$ | $2x + 10 = 16$ |
| $2x + 9 = 15$ | |
| $x + 9 = 15 − x$ | $2x + 8 = 14$ |
| $9 = 15 − 2x$ | $2x + 7 = 13$ |
| $9 − x = 15 − 3x$ | $2x + 6 = 12$ |
| The rule for this arm is _____ each time | The rule for this arm is _____ each time |
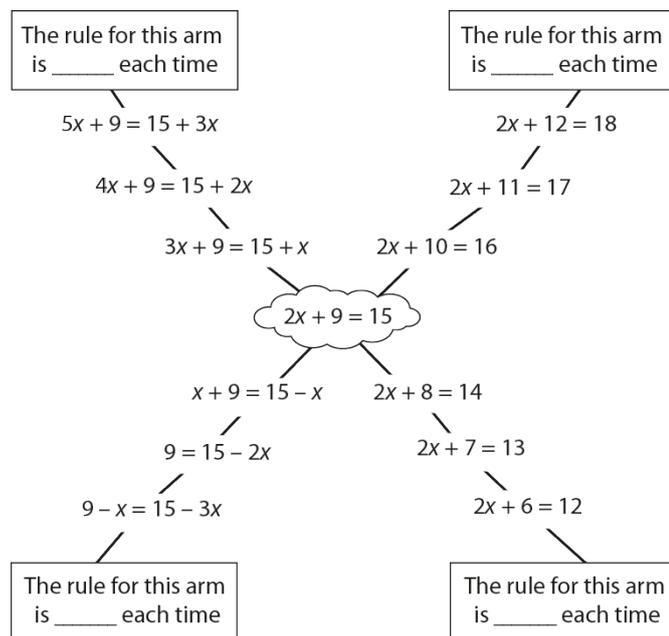
*Each arm of the diagram is constructed using a rule that is repeated at each step moving out from the centre.*

a) *Which arm has the rule 'add one to each side each time'?*

b) *Write down the rules for the other arms.*

**V** The use of a spider diagram constructed according to given rules can give students an opportunity to understand the impact of making a change to an equation while maintaining the equilibrium. This is not intended to be an efficient strategy for solving equations; rather the intention is to offer insight into the structures that underpin the process of 'performing the same operation on each side'.

A key realisation is that every equation in the diagram has the same solution, since each change that has been made maintains the equilibrium.

**PD** Once students have made sense of this spider diagram, and identified the rules used, you might follow this with a task in which they create their own spider diagram following the same rules but with a different starting point, or offer a starting point that is not the centre of the spider, for example:

c) *The solution to $2x + 9 = 15$ is $x = 3$. Is this also the solution to the other equations that have been created?*

| Double each side each time |  | Add $3x$ to each side each time |
|---|---|---|



$3x + 5 = 7 + 2x$

| Subtract 5 from each side each time |  | Add $x$ to each side each time |
|---|---|---|
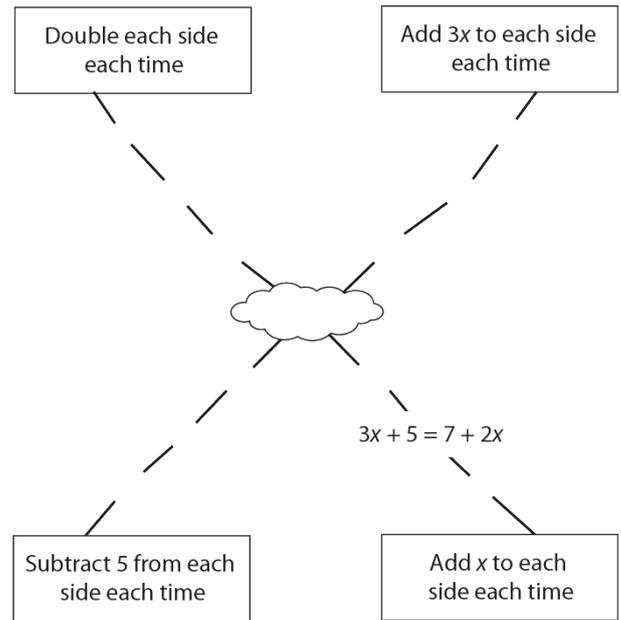
Consider the benefits of these next steps – how would you follow this task?

*Example 4:*

*The solution to the equation $24x + 5 = 17x + 61$ is $x = 8$.*

*Use this to find the value of x when:*

a) *the 24 and 17 are doubled to give the equation* **48***x* + 5 = **34***x* + 61

b) *the 5 and 61 are doubled to give* $24x +$ **10** $= 17x +$ **122**

c) *the entire equation is doubled to give* **48***x* + **10** = **34***x* + **122**.

**V** *Example 4* is designed to make explicit some of the manipulations that will and will not maintain equilibrium in an equation. Students should be discouraged from rushing in to solve the equation by manipulating it; rather, they should be given the opportunity to reason and make predictions about the impact of making the changes described.

It may be beneficial to offer part a) and discuss this, before then offering part b) and again discussing students' predictions and reasoning, followed by part c), rather than offering all three as a single exercise.

**PD** How do you think students will answer each of these questions? Predict the responses that will arise and the reasoning that might accompany them. What strategies and structures will you use in your classroom to help students share their reasoning?

A similar task can be created by adding a constant value on to each term (for example, replacing part a) with two is added onto the 24 and 17 to give the equation **26***x* + 5 = **19***x* + 61). This task offers a set of

transformations that maintain the equilibrium but may offer less opportunity for students to reason. Do you feel that an additive version of the task is worthwhile?

| | |
|---|---|
| *Example 5:*<br><br>*Given that $4x + 12 = 10x + 2$, which of these are true and which are false? Explain how you know.*<br><br>a) $4x + 13 = 10x + 3$<br><br>b) $8x + 24 = 20x + 4$<br><br>c) $4 + 12 = 10 + 2$<br><br>d) $3x + 12 = 10x + 2 - x$<br><br>e) $2x + 6 = 5x + 1$<br><br>f) $13 = 6x + 3$<br><br>g) $8x + 12 = 20x + 2$<br><br>h) $4x = 10x + 2 - 12$ | **V** There are only two equations here – parts c) and g) – that are not a correct transformation of the given equation. Some of the transformations offered – parts d) and h) – have not been simplified, and this may be a useful discussion point for students.<br><br>Although the intention of this task is not to focus on processes for solving an equation, a useful prompt might be to ask which of the valid transformations students think moves closer to a solution and why.<br><br>**PD** The two non-examples used in this example are chosen to highlight particular misconceptions. In part c), the '$x$'s have been 'taken away' (they have been removed from the equation rather than being subtracted). In part g), the coefficient of the $x$ has been doubled on each side, but the constant term has not been changed.<br><br>Consider other misconceptions that students might have. What non-examples might you add to this set in order to expose these misconceptions? |

Understand the manipulations that will efficiently work towards solving an equation.

*Example 6:*

*Look at the spider diagram.*



The rule is subtract $x$ from each side each time

The rule is add $x$ to each side each time

$8x + 15 = 4x + 21$

$3x + 15 = 21 - x$

$4x + 15 = 21$

$7x + 15 = 3x + 21$

$5x + 15 = x + 21$

$6x + 15 = 2x + 21$

$6x + 10 = 2x + 16$

$6x + 14 = 2x + 20$

$6x + 5 = 2x + 11$

$6x + 13 = 2x + 19$

$6x = 2x + 6$

$6x + 12 = 2x + 18$

$6x - 5 = 2x + 1$

The rule is subtract 5 from each side each time

The rule is subtract 1 from each side each time
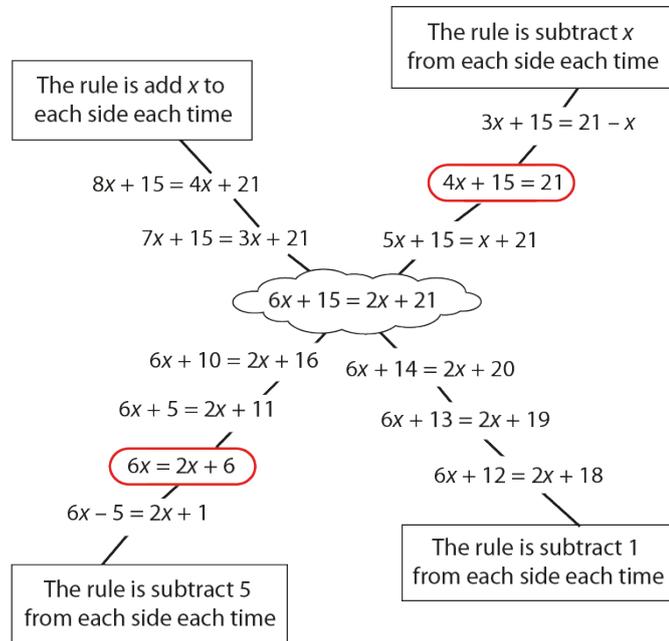
*Explain why the two highlighted transformations are useful in finding a solution to the equation $6x + 15 = 2x + 21$.*

From previous use of these spider diagrams (in *Example 3*), students should know that all of the equations in the diagram have the same solution. The focus for *Example 6* is to draw attention to particular transformations that move closer to a solution to the equation. A key understanding here is that removing one of the terms of the equation makes finding a solution more straightforward, and it is this that should be a focus of this task.

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*Example 7:*

*Parts (i)–(vii) are all correct transformations for $11d - 9 = 7d + 3$.*

a) *Which ones are helpful in finding a solution?*

b) *Which would you use to find the value of d? Explain why.*

   (i)   $d - 9 = 3 - 3d$

   (ii)  $-9 = 3 - 4d$

   (iii) $11d = 7d + 12$

   (iv)  $10d - 9 = 6d + 3$

   (v)   $4d - 9 = 3$

   (vi)  $11d - 12 = 7d$

   (vii) $0 = 12 - 4d$

**V** In *Example 7*, students are offered a range of correct manipulations, allowing them to focus on making a decision about which manipulation(s) will move them closer to solving the equation.

A useful discussion might be had around what transformation has taken place to change the original equation to each of the examples (with the exception of part (vii), each example is the result of just one change), but the key idea here is that some of these transformations – parts (ii), (iii), (v), (vi) and perhaps (vii) – reduce the number of terms in the equation while maintaining the equality, and so move closer to a solution.

**PD** Do you have a preference for the order in which equations like this are solved? Do you teach students to manipulate the constant terms first – as in parts (iii) and (vi) – or to deal with the unknown first – as in parts (ii) and

(v)? Do you consider the 'larger' term first or the 'smaller'? Why do you teach this particular process?

*Example 8:*

*Use the given information to fill in each gap with a number or an expression so that the equations balance.*

*a)*



| $n = 2$ |

| $11n =$ ___ |

| $11n + 7 =$ ___ |

| $13n + 7 =$ ___ |

*b)*

| $t =$ ___ |

| $3t =$ ___ |

| $3t - 1 = 23$ |

| $5t - 1 =$ ___ |

*c)*

| $m =$ ___ |

| $5m =$ ___ |

| $5m + 6 =$ ___ |

| $2m + 6 = 61 - 3m$ |

In *Example 8,* students are taken through a particular set of steps, and their attention should be drawn to the way in which it is possible to move 'backwards' through these by choosing efficient transformations, meaning that the unknown is on one side of the equals sign and the constant term on the other.

The focus here is on what has changed from one step to the next, and so undoing this in order to transform the equation into a simpler equivalent equation.

*Example 9:*

*Solve these equations.*

a) (i) $5x + 8 = 43$

   (ii) $7x + 8 = 2x + 43$

   (iii) $x + 8 = 43 - 4x$

b) (i) $2 - x = 3x - 12$

   (ii) $2 - x = 12 - 3x$

   (iii) $x - 2 = -3x - 12$

c) (i) $x + 73 = 5x + 5$

   (ii) $x + 7.3 = 5x + 0.5$

   (iii) $\frac{1}{2}x + 7.3 = 5x + 0.5$

**V** *Example 9* provides an opportunity for students to practise solving equations with the unknown value on each side.

In part a), equations (ii) and (iii) are transformations of equation (i). This may give students a chance to discuss reducing the complexity of an equation by ensuring that the unknown is on just one side.

Part b) offers practice at making decisions about which term of the equation to 'move' to make solving quick and efficient (for example, in part (i), students might start by adding $x$ to each side, while in parts (ii) and (iii) they might add $3x$ to each side). This task

| | may provide different routes to reach a solution for students to discuss and unpick. |
|---|---|
| | Part c) introduces fractional and decimal elements to the equation for students to work with. This may allow them to further generalise the process of solving an equation by working with more complex numbers. |

## Weblinks

[1] NCETM primary mastery professional development materials
https://www.ncetm.org.uk/resources/50639

[2] NCETM primary assessment materials
https://www.ncetm.org.uk/resources/46689

[3] ICCAMS Maths
http://iccams-maths.org/

[4] GeoGebra file for the representation of $3x + 11 = 5x + 1$
https://www.geogebra.org/m/ej4x9mw4