9 Sequences, functions and graphs

Mastery Professional Development

9.5 Exploring trigonometric functions

Guidance document | Key Stage 4

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Click the heading to move to that page. Please note that these materials are principally for professional development purposes. Unlike a textbook scheme they are not designed to be directly lifted and used as teaching materials. The materials can support teachers to develop their subject and pedagogical knowledge and so help to improve mathematics teaching in combination with other high-quality resources, such as textbooks.

Making connections

Building on the Key Stage 3 mastery Professional Development (PD) materials, the NCETM has identified a set of five 'mathematical themes' within Key Stage 4 mathematics that bring together a group of 'core concepts'.

The third of the Key Stage 4 themes (the ninth of the themes in the suite of Secondary Mastery Materials) is *Sequences, functions and graphs,* which covers the following interconnected core concepts:

9.1 Exploring linear equations and inequalities

9.2 Exploring non-linear sequences

9.3 Exploring quadratic equations, inequalities and graphs

9.4 Exploring functions

9.5 Exploring trigonometric functions

This guidance document breaks down core concept *9.5 Exploring trigonometric functions* into two statements of **knowledge, skills and understanding:**

9.5 Exploring trigonometric functions

- 9.5.1 Explore trigonometric functions with arguments in degrees
- 9.5.2 Transform trigonometric functions

Then, for each of these statements of knowledge, skills and understanding, we offer a set of key ideas to help guide teacher planning:

- 9.5.1 Explore trigonometric functions with arguments in degrees
  - 9.5.1.1 Revisit the unit circle to explore the sine, cosine and tangent of angles greater than 90 degrees
  - 9.5.1.2 Understand the features of the sine, cosine and tangent functions for $0° \leq \theta \leq 360°$
  - 9.5.1.3 Understand the graphical features of the sine, cosine and tangent functions with arguments in degrees
- 9.5.2 Transform trigonometric functions
  - 9.5.2.1 Recognise the effect of transformations applied to the domain of a trigonometric function, such as $y = f(x + a)$, $y = f(-x)$ and $y = f(ax)$
  - 9.5.2.2 Recognise the effect of transformations applied to the range of a trigonometric function, such as $y = f(x) + a$, $y = -f(x)$ and $y = af(x)$

Overview

In this core concept, students have an opportunity to think functionally about existing mathematical knowledge. Through exploration of the familiar representation of the unit circle, students' established knowledge of trigonometry is revised and extended alongside their emergent functional thinking.

Students will have already encountered the trigonometric functions (sine, cosine and tangent), but these ideas will have primarily been explored in a different way. In the Key Stage 3 PD materials, trigonometric functions were introduced through the unit circle but with a focus on angles between 0° and 90°. The emphasis was on how this representation could be used to visualise the relationships between sides in right-angled triangles, leading to the use of trigonometric ratios to solve problems such as finding the values of missing sides or angles.

Here, trigonometric functions are once again explored through the lens of the unit circle, but with an emphasis on representing these relationships graphically. Every attempt has been made to use pictorial representations to convey how the rotation of a point about the unit circle can be used to generate each of these functions. However, it remains that these are continuous functions and that is a challenging property to convey in static images. To fully understand the mathematical structures inherent in trigonometric functions, it is helpful for students to experience explanations and explorations accompanied by clear and understandable visuals. It is therefore recommended that teachers use dynamic geometry and graphing software when exploring these functions, both for supporting students to develop a deep and connected understanding, and also for consolidating and extending their own subject knowledge.

Moving from considering trigonometric functions solely for angles in right-angled triangles – which can easily be drawn and visualised – is a significant step for students. Spend time ensuring students understand angles as a measure of turn, and appreciate the potentially infinite nature of that turn. It is also a step that sits across different themes within these materials. Trigonometry in non-right-angled triangles is explored in detail in '*11.3 Trigonometry'*, and an appreciation of the shape of each of the trigonometric functions is an essential step in understanding why, for example, an acute and obtuse angle can have the same sine value. Consider the order in which students encounter the various strands of new learning at Key Stage 4 and how new learning (such as of the sine or cosine rule) can simultaneously offer an opportunity for consolidation of existing learning (such as the graphs of the sine and cosine functions).

The key ideas in this core concept provide a structure to enable students to gain insight into the particular properties of the trigonometric functions, while also drawing their attention to the broader properties of functions in general. This core concept is also likely to be the first time that students will have encountered a cyclic function. The repeating nature of such functions offers a useful context with which to further explore the effects of transformations on graphs. This will enable them to develop a deep and connected understanding, while also ensuring that those students who choose to study mathematics beyond Key Stage 4 have a secure basis on which to build future learning.

Prior learning

Students will continue to build on their understanding of graphical representations from '*9.4 Exploring functions'*. In the context of trigonometry, students' awareness of the infinite nature of the x- and y-axes is particularly important, as it is the first time they will have encountered a cyclic function that repeats infinitely.

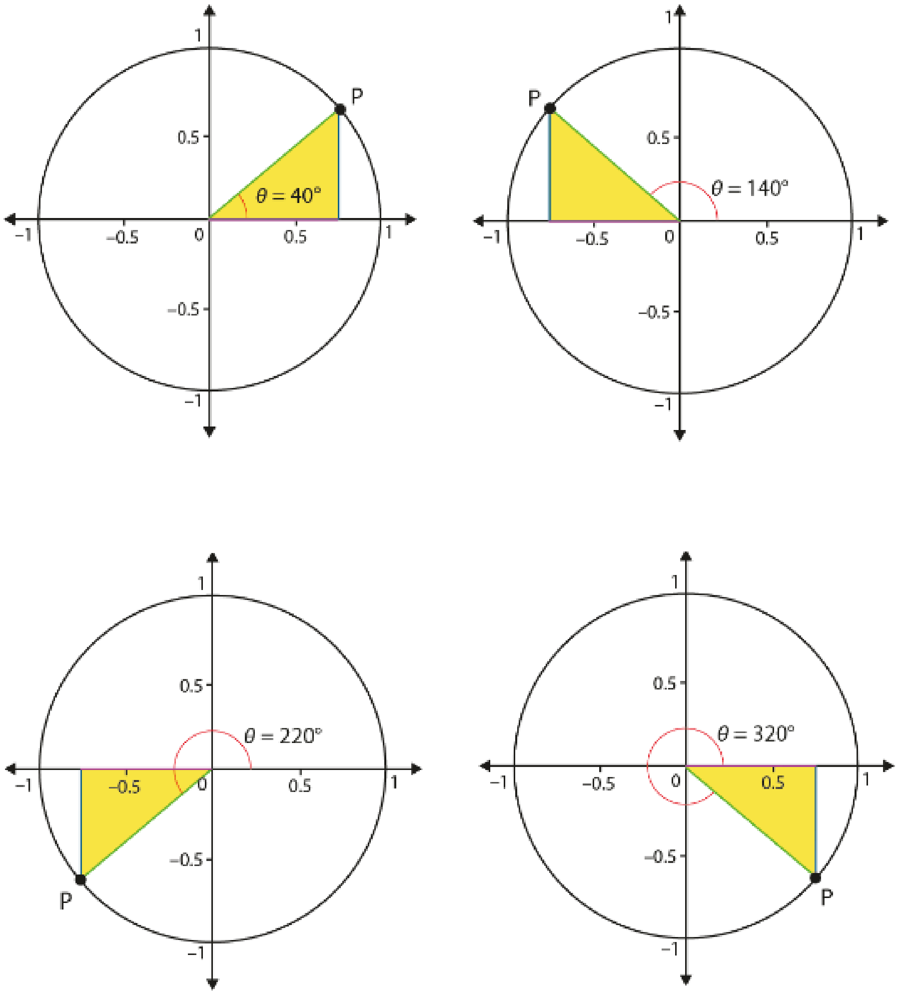
Trigonometry is an area of mathematics that sits across several different themes in the NCETM PD materials. It may be that students were introduced to the trigonometric ratios using the unit circle in the Key Stage 3 PD materials document '*3.2 Trigonometry'*. Regardless, it is likely that the bulk of students' experience with trigonometry has been drawing on their prior knowledge of ratio (in terms of the relationship between sides in right angled triangles) or algebra (in terms of manipulating and solving equations to find missing values). Thinking about these relationships in a more functional sense might therefore, to many students, feel like entirely new learning.

They should have begun to feel familiar with the idea of functions as objects in their own right, and started to extend their understanding of transformations to include graphical transformations. The trigonometric functions offer a further context with which to embed this knowledge.

The Key Stage 3 PD materials documents '*3.2 Trigonometry', '4.2 Graphical representations'* and '*6.3 Transforming shapes'* all explore the prior knowledge required for this core concept in more depth.

Checking prior learning

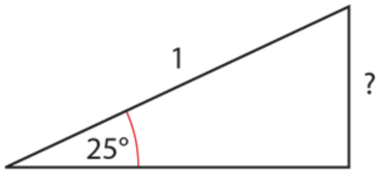
The following activities from the NCETM Secondary Assessment materials, Checkpoints and/or Key Stage 3 PD materials offer a sample of useful ideas for assessment, which you can use in your classes to check understanding of prior learning.

Reference	Activity
Key Stage 3 PD materials document '*3.2 Trigonometry'*, Key idea 3.2.1.1, *Example 5*	What is the same and what is different about these triangles? Explain how you know. 

Key Stage 3 PD materials document '*3.2 Trigonometry'*, Key idea 3.2.1.3, *Example 3*	Using the relevant values in the table below (or using your calculator), find the missing side lengths.

Angle	Sine	Cosine	Tangent
10	0.1736	0.9848	0.1763
15	0.2588	0.9659	0.2679
20	0.3420	0.9397	0.3640
25	0.4226	0.9063	0.4663
30	0.5000	0.8660	0.5774
35	0.5736	0.8192	0.7002
40	0.6428	0.7660	0.8391
45	0.7071	0.7071	1
50	0.7660	0.6428	1.1918
55	0.8192	0.5736	1.4281
60	0.8660	0.5000	1.7321
65	0.9063	0.4226	2.1445
70	0.9397	0.3420	2.7475
75	0.9659	0.2588	3.7321
80	0.9848	0.1736	5.6713
85	0.9962	0.0872	11.4301
90	1	0	∞

a)



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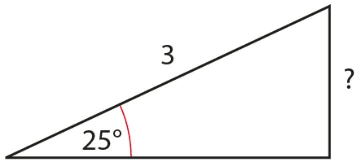

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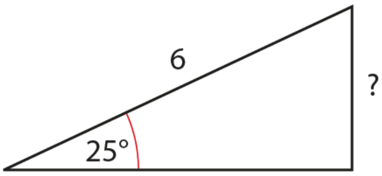

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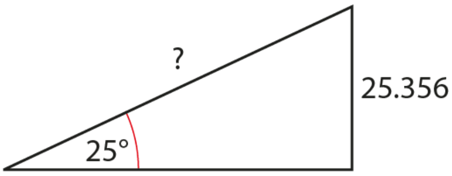
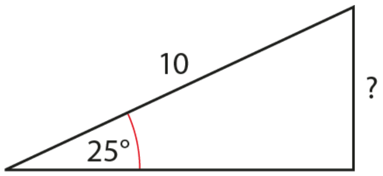


	e)  (Diagrams are not drawn to scale)

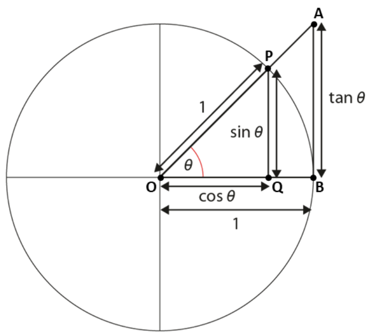
Key vocabulary

Key terms used in Key Stage 3 materials

- trigonometric functions (sine, cosine, tangent)

The NCETM's mathematics glossary for teachers in Key Stages 1 to 3 can be found here.

Key terms introduced in the Key Stage 4 materials

Term	Explanation
domain	The domain of a function, $f(x)$ is the set of arguments, x, for which that function is defined.
function	A function is a relationship between a set of inputs and a set of outputs, with the property that each input in the first set (the domain) maps to a single associated output in the second (the range). Note that the converse need not be true, for example when $f(x) = x^2$ and the domain is the real numbers.
trigonometric functions (sine, cosine, tangent)	Functions of angles. The main trigonometric functions are cosine, sine and tangent. Other functions are reciprocals of these. Trigonometric functions (also called the 'circular functions') are functions of an angle. They relate the angles of a triangle to the lengths of its sides. The most familiar trigonometric functions are the sine, cosine and tangent in the context of the standard unit circle with radius 1 unit, where a triangle is formed by a ray originating at the origin and making some angle with the x-axis; the sine of the angle gives the length of the y-component (rise) of the triangle, the cosine gives the length of the x-component (run), and the tangent function gives the slope (y-component divided by the x-component). 

	Trigonometric functions are commonly defined as ratios of two sides of a right-angled triangle containing the angle. They can equivalently be defined as the lengths of various line segments from a unit circle: $\sin(\theta) = \frac{PQ}{OP} = PQ$ $\cos(\theta) = \frac{OQ}{OP} = OQ$ $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} = \frac{PQ}{OQ}$
range	The range of a function, $f(x)$ is the set of values that $f(x)$ can produce for the given domain.

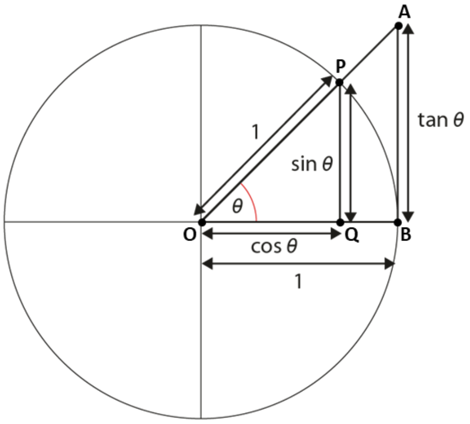
Key ideas

In the following list of the key ideas for this core concept, selected key ideas are marked with a . These key ideas are expanded and exemplified in the next section – click the symbol to be taken direct to the relevant exemplifications. Within these exemplifications, we explain some of the common difficulties and misconceptions, provide examples of possible student tasks and teaching approaches and offer prompts to support professional development and collaborative planning.

9.5.1 Explore trigonometric functions

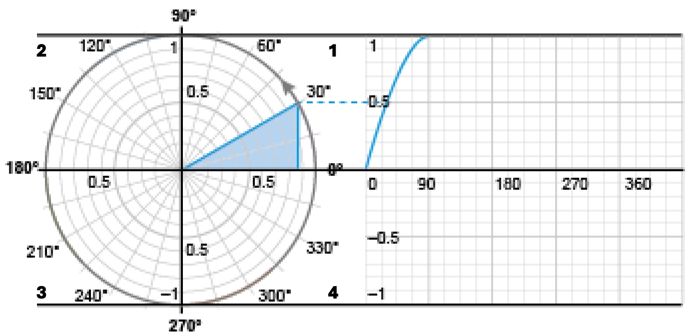
Students need an appreciation of trigonometric functions as they relate to the size of particular lines formed on the unit circle This was introduced in the Key Stage 3 PD materials document *'3.2 Trigonometry*'. However, in that early stage of their learning, students are likely to have mainly considered trigonometric functions as relationships between the angles and sides of right-angled triangles. For example, they might be able to describe how $\sin(\theta)$ is equivalent to the ratio $\frac{opposite}{hypotenuse}$.

While they may have seen an image of the unit circle (below) they may not have fully appreciated its role in understanding the trigonometric functions.

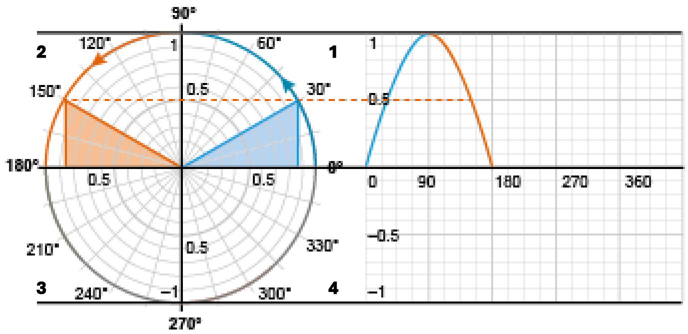


A more complete understanding of trigonometric functions involves considering them as mappings from an angle to a value. It may be the first time that students deeply explore the covariation that occurs more generally in functional relationships. That is, that one variable (the ratio of the sides, also known as the value of $\sin(\theta)$) changes instantaneously as the other variable (the angle of turn, also known as the value of θ) changes. This mapping is not entirely multiplicative, and the way the output changes compared to the input is not always intuitive. The unit circle provides a representation of that relationship that can demonstrate why, for example, $\sin(60°)$ is not double $\sin(30°)$.

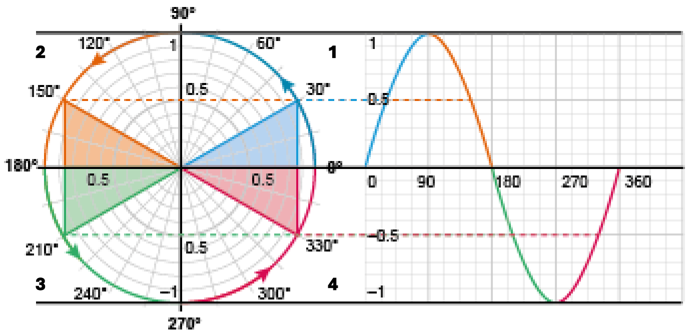
Throughout Key Stage 3, it is likely that the domain of the trigonometric functions has exclusively been angles between $0°$ and $90°$, as demonstrated for $\sin(\theta)$ here.



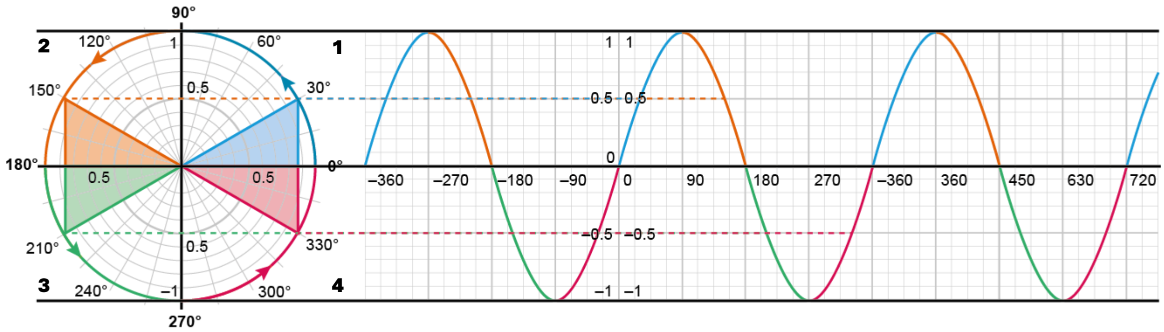
At Key Stage 4, students should begin to consider angles greater than $90°$, connecting this to the four congruent triangles shown. When the domain is between $0°$ and $180°$, this can still connect to work on missing angles in triangles.



Students should then be made aware that, in one complete revolution of the unit circle, there are two angles that result in the same height above 0, and two angles that result in that same height below 0.



Building from an initial exploration of angles up to $360°$, students will need to begin associate the repeating cyclic nature of a trigonometric function its graphical representation. They should relate positions on the unit circle to their corresponding positions of the graph of $y = \sin(\theta)$ below:



The static images above highlight key features of the relationship, but this relationship can be demonstrated more clearly using dynamic geometry software, which can help students to connect the cyclical nature of the movement around the unit circle to the cyclical nature of the function. These relationships can also then be shown at other points and related to associated relationships on the graphs of $y = \cos(x)$ and $y = \tan(x)$. This can help to deepen understanding both of the trigonometric functions and of functions more generally, preparing students for mathematics beyond GCSE.

9.5.1.1 Revisit the unit circle to explore the sine, cosine and tangent of angles greater than 90 degrees

9.5.1.2 Understand the features of the sine, cosine and tangent functions for $0° \leq \theta \leq 360°$

9.5.1.3 Understand the graphical features of the sine, cosine and tangent functions with arguments in degrees

9.5.2 Transform trigonometric functions

Students will already have begun to encounter the effect of various transformations on functions. In '*9.4 Exploring functions'*, the emphasis was on connecting the algebraic and graphical representations; the key ideas were separated into different transformations. For example, the exemplified key idea 9.4.3.5

explored translations in depth. Here, instead, they are grouped according to the part of the function that is affected by the transformation – that is, whether the transformation is applied to the domain or the range. This alternative paradigm offers students another ‘way in’ to help them further conceptualise transformations in the context of functions. The intention is to support students to build a deep and connected understanding, rather than remembering the effects of transformations as a series of rote-learnt and easily forgotten statements.

Trigonometric graphs, with their cyclical nature, offer an ideal context to further deepen students’ understanding both of transformations and of the trigonometric functions themselves. Students should recognise that, across a given domain, the range of values for each trigonometric function will repeat every 360° (for sine and cosine) or 180° (for tangent). As such, there are some transformations that can appear to have no effect on the graph, such as $y = \sin(x + 360)$ or $y = \tan(x + 180)$. The repetition of the sine and cosine curves, alongside the limited range of y-values that they map to $(-1 \leq y \leq 1)$, means that these two functions in particular offer a useful visual context with which to explore the effects of multiplication on the domain and range. Considering the variant and invariant features after each transformation will provide students with a structure to think more deeply about the properties of each function.

9.5.2.1 Recognise the effect of transformations applied to the domain of a trigonometric function, such as $y = f(x + a)$, $y = f(-x)$ and $y = f(ax)$

9.5.2.2 Recognise the effect of transformations applied to the range of a trigonometric function, such as $y = f(x) + a$, $y = -f(x)$ and $y = af(x)$

Exemplified key ideas

In this section, we exemplify the common difficulties and misconceptions that students might have and include elements of what teaching for mastery may look like. We provide examples of possible student tasks and teaching approaches (in italics in the left column), together with ideas and prompts to support professional development and collaborative planning (in the right column).

The thinking behind each example is made explicit, with particular attention drawn to:

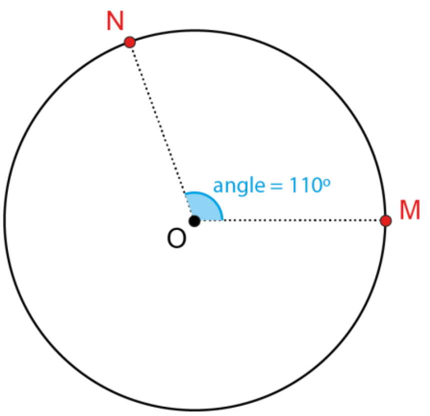
Deepening	How this example might be used for **deepening** all students' understanding of the structure of the mathematics.
Language	Suggestions for how considered use of **language** can help students to understand the structure of the mathematics.
Representations	Suggestions for key **representation(s)** that support students in developing conceptual understanding as well as procedural fluency.
Variation	How **variation** in an example draws students' attention to the key ideas, helping them to appreciate the important mathematical structures and relationships.

In addition, questions and prompts that may be used to support a professional development session are included for some examples within each exemplified key idea.

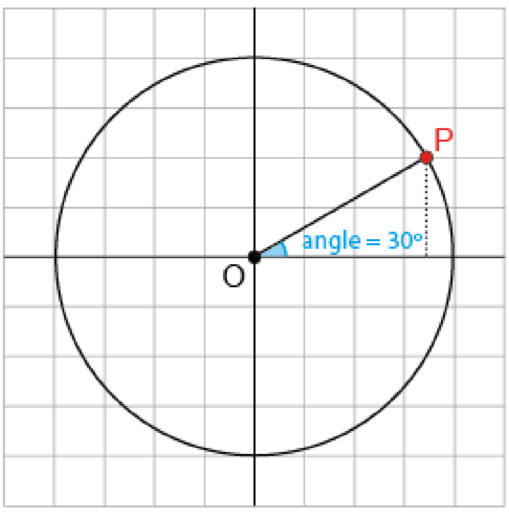
These are indicated by this symbol.

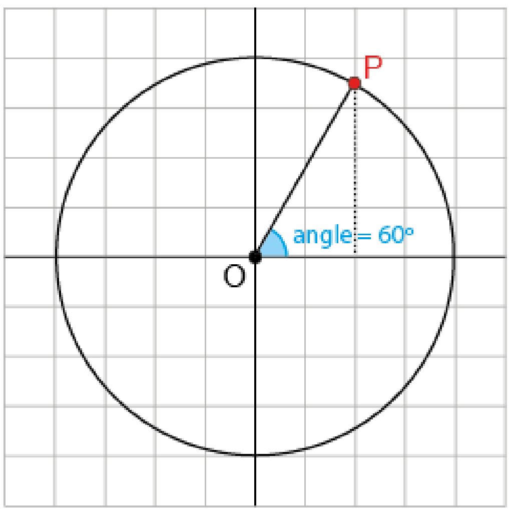
9.5.1.3 Understand the graphical features of the sine, cosine and tangent functions with arguments in degrees

Common difficulties and misconceptions
The unit circle was introduced as a representation for trigonometry in the Key Stage 3 PD materials document '*3.2 Trigonometry*'. Students should have been supported to see trigonometry as more than just a relationship between sides and angles but also as relationships on the unit circle. This supports deeper understanding of the functional relationships inherent in trigonometry and prepares students to see and understand the periodic nature of trigonometric functions. In this key idea, we further develop this representation of the function, using graphs on a Cartesian coordinate grid. The shift to understanding the periodic nature of the graphs of trigonometric functions – for example, understanding that x can take a value greater than 180° in the graph of $y = \sin x$ – can still be challenging. Students who have not been introduced to the unit circle as a representation to give access to trigonometry, or who found it difficult to access during Key Stage 3, will need time, and tasks designed to make explicit the connection between their understanding of trigonometric ratios and the unit circle.

Students need to	Guidance, discussion points and prompts
Understand it is possible to turn an angle greater than 360° *Example 1:* *Graham, Miles and Ray all start facing the same direction, then they spin on the spot, turning in the same direction.*  ***Graham*** ***Miles*** ***Ray*** *Graham spins 810° then stops.* *Miles spins 720° then stops.* *Ray spins 1100° then stops.* *a) Who is shown in picture i? And picture ii?* *(i)*  *(ii)*  *b) Sketch the position of the third person after they've completed their turn.* *c) In which direction are they all turning?*	*Example 1* gives a concrete, if slightly odd, **representation** to remind students of angles greater than 360° and how they might visualise them. Students may be aware of angles as a measure of turn and know that angles can be greater than a whole turn. However, it is likely that their most recent experience of angles in geometry is between 0° and 360°. Note the **variation** here: the values have been chosen so that Miles spins a multiple of 360° and Graham's turn is then 90° more than this. Awareness of the use of multiples of 360° to support the visualisation and to quantify the absolute turn is key for students here. The use of 1100° for Ray then shifts this on to the third multiple of 360 and reinforces the repeating nature of the measure of turn. This is revisited in *Example 2* in a more formal context.
Example 2: *A point travels anticlockwise around the circumference of a circle. The centre of the circle is at O.* *The point starts at position M and, some time later, is at N.*  *Lizi says, 'The point has travelled 110° around the circumference of the circle.'*	*Example 2* offers a more formal perspective on the content of *Example 1*, this time using the **representation** of a circle. Students could regard the angle created as being between two radii, but this is a static image, and it is important to ensure students have understood the dynamic context of points travelling around the circumference. Again, the emphasis is on visualising turns of greater than 360°, using multiples of 360 to support this visualisation and to give a numerical value to the resultant turn. The angle 110° is chosen as a simple number to work with and adjust using multiples of 360. This allows students to focus on the reasoning in an unfamiliar context rather than on calculation strategies. There are opportunities for **deepening** students' thinking around the domain; for example, by recognising that for some functions this can be infinite. This can be facilitated through considering a constant and regular rotation of a point on the circumference of a unit circle. Later examples will build on Key Stage 3 PD materials document 3.2, and rely on thinking of sin, cos and tan as functions in their own right, rather than as ratios between sides in a triangle. The key idea to establish is that the repeating, cyclic nature of

Steve says, 'The point has travelled 470° around the circumference of the circle.' *Beth says, 'The point has travelled 3710° around the circumference of the circle.'* *a) Who do you agree with? Why?* *b) How sure are you that you are correct? Justify your answer.*	the motion of the point on the circumference is reflected in the repeating, cyclic nature of the trigonometric functions.
Connect the graphs of trigonometric functions to the unit circle *Example 3:* *In the images below, point P moves anticlockwise around the circumference of the circle.* *A snapshot is taken when P has travelled through 30°, and another when P has travelled through 60°. These diagrams are drawn accurately.* *a) The angle, shown below, has doubled from 30° to 60°. Has the height of P above the horizontal line doubled? About how much has it increased by?* *b) Imagine that the original angle is trebled. Has the height of P above the horizontal line trebled? About how much has it increased by?* *c) Imagine that the original angle is multiplied by four or five. What happens to the height of P above the horizontal line?* *d) Describe what happens in general as the angle increases from 0°.*	*Example 3* uses the sine function as an example to explore the way in which the unit circle can be a helpful **representation** of a trigonometric function. We could also study cosine by examining the horizontal distance from the origin, or the tangent by examining the length of a line that meets $x = 1$. This is explored in the later examples. By considering the height of the points above the x-axis at particular angles, the non-linear nature of the functions is clear. The **variation** in the task design draws students' attention to how the two variables (angle and vertical height of P) do not both change by the same proportion. Parts c and d then give a context to revisit the cyclic nature of the sine function. *Examples 3* to *8* build conceptual understanding of the shapes of the three trigonometric graphs. Consider who you might work through these examples with. Are they all suited to the classroom? Are these tasks you might work on as a department? Reflect on how teachers might find them helpful. As well as having potential use in the classroom, these examples could support the subject knowledge of your department. This is particularly true of any teachers who are less confident with topics that are typically taught towards the end of Key Stage 4, perhaps due to their own training or experience.

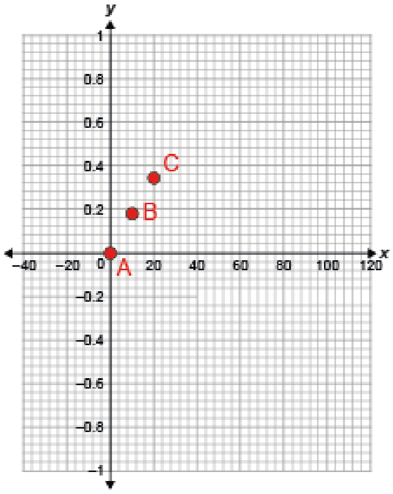




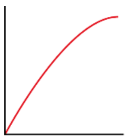
Example 4:

Vic draws a graph of the height of point P (from Example 3) at different angles as it travels anticlockwise around the circumference of a circle.

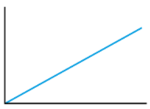
In this graph he's plotted the height for 0°, 10° and 20°:



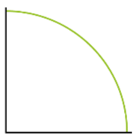
a) *Which of the graphs below do you think is most like the graph Vic will draw when he plots the rest of the heights from 0° to 90°?*



Graph X



Graph Y



Graph Z

The heights Vic measures at different angles are shown in the table below this example.

b) *Plot these points on the grid (also below this example) to show the change in the height of the line from 0 to 90°. The first three (A, B and C) are already plotted.*

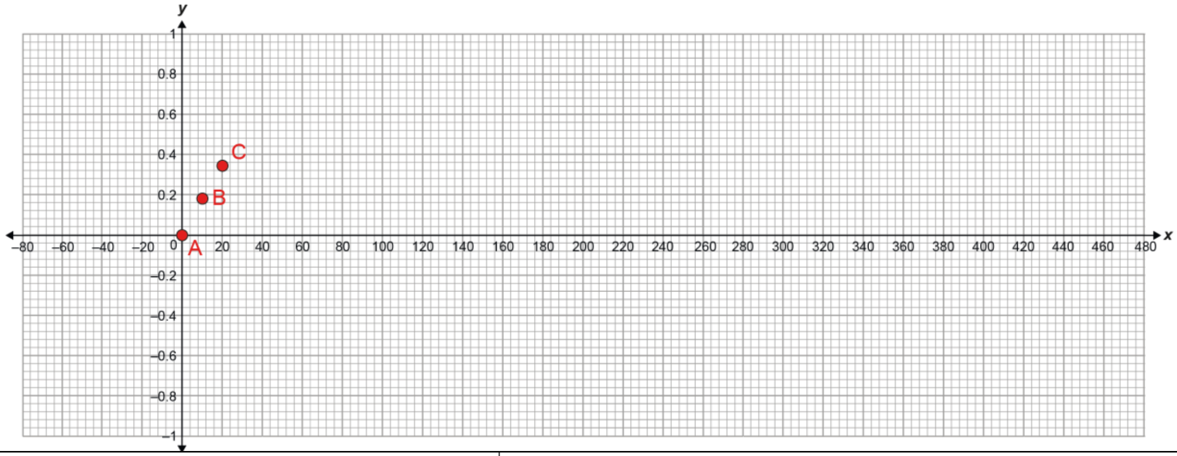
c) *Using the information he already has, Vic realises that he can use symmetry to plot points for all angles up to 180°. Explain how he can do this.*

Example 4 also explores the sine function but focuses on the graphical **representation**. If students are still getting to grips with the unit circle, which here is just described in words, then it may be worth showing them a dynamic image of a point moving around the circumference alongside this example. The intention is that students have an opportunity to work between these different representations and make connections, noting key points and reasoning about how the value of the function varies between those points.

Throughout, students' focus shifts between considering individual points (the introductory text, part b) and the overall shape of the graph (parts a and c). The intention of this is **deepening** students' understanding of the nature of the trigonometric functions, so that they are considering the overall relationship between the two variables rather than simply plotting points.

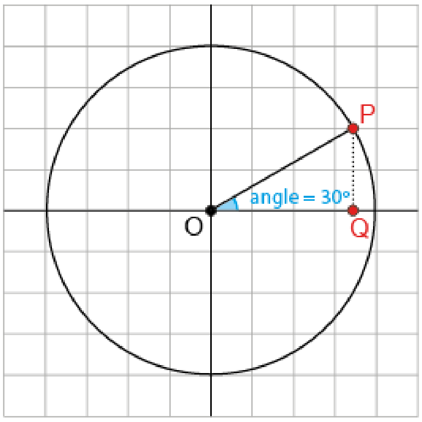
In part b, a version of a trigonometric table is offered for students to plot the points of $f(\theta) = sin(\theta)$ for values of θ between 0° and 90°. Discuss your early experiences of learning this with your colleagues. Did you use trigonometric tables? Did you use calculators? Did you use both? Ask colleagues who had experience of just one of these tools to reflect on their early and current understanding of trigonometry. Is there a difference in how they conceptualise trigonometric functions?

Angle	*0*	*10*	*20*	*30*	*40*	*50*	*60*	*70*	*80*	*90*
Height	*0*	*0.17*	*0.34*	*0.5*	*0.64*	*0.77*	*0.87*	*0.94*	*0.98*	*1*



Example 5:

The circle below has a radius of one unit.

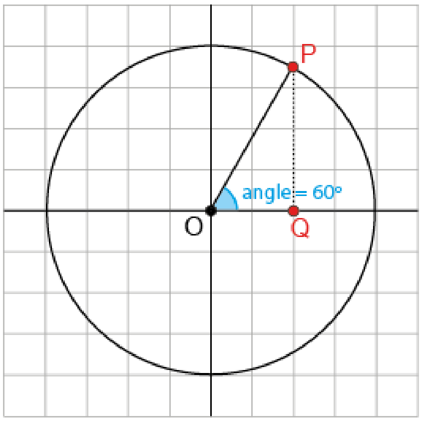


a) *What is the distance OP?*

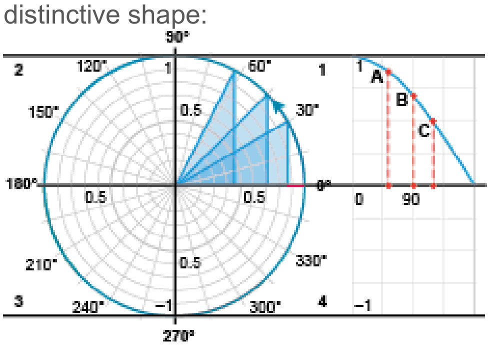
b) *How could someone calculate the distance OQ?*

c) *How could someone calculate the distance PQ?*

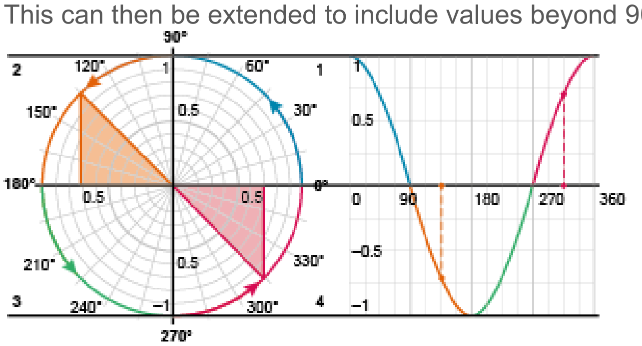
Point Q remains on the x-axis and QC is always a vertical line. The circle below shows this when the point has moved.



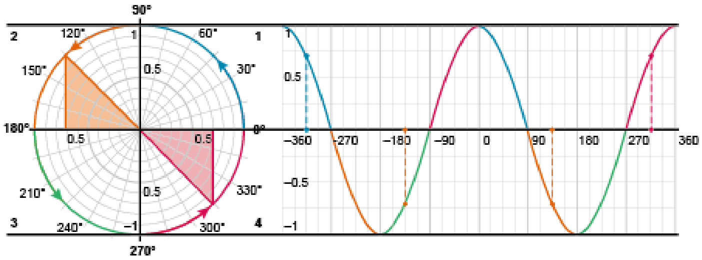
Recognising how graphical **representations** of sine, cosine and tangent can be created using the unit circle is a key understanding when working with trigonometric relationships as functions. *Example 5* places the emphasis on the cosine function, and so students should be considering the length along the x-axis (OQ) and the way that varies as point P rotates. This is perhaps less easy to imagine using static images than the height above the x-axis for the sine function. Software can demonstrate dynamically the way that the cosine changes as the point moves round the circle or be used to evaluate and plot some key points. It is important that students have a chance to familiarise themselves and think about this context before the graph of the functions is shared. You might then like to share an image such as this, to demonstrate which lengths on the unit circle create its distinctive shape:



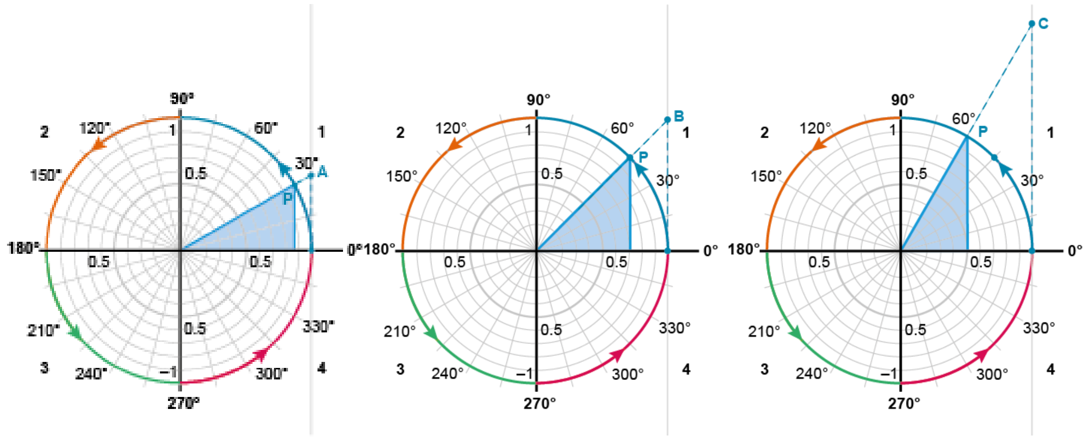
This can then be extended to include values beyond 90°:



Identifying particular points in the different representations, and noticing how one is connected to the other, is key.

d) Which of the three distances (OP, OQ and PQ) has changed, and which has stayed the same? *e) If we call the angle ‘θ’, describe how to find OP, OQ, and PQ for any angle.* *f) Describe where you will always see $\sin(\theta)$ and $\cos(\theta)$ on the diagram.* *Imagine the point moving anticlockwise around the circumference of the circle.* *g) What is cosine of θ at 0°? 90°? 180°? 270°?* *h) Use the information in part g to sketch what the graph of the cosine of θ looks like between 0° and 360°* *i) What is cosine of θ at 360°? 540°? 720°? What does this mean for the graph of the function?*	Students are likely to need time and prompting to make these connections, particularly when considering the cyclic nature of the trigonometric functions:  Working between different representations like this can help with **deepening** students’ conceptual understanding of a topic. Educational research suggests that students who can move fluently between representations have a more secure understanding of the concepts being represented. It is therefore advisable to not only develop students’ competence with one representation, but also to support them to move between multiple representations and appreciate the links between them. They should be guided to see the similarities and differences between the different representations. How do you approach representations in this topic? How do you support students to link the visual features of one representation to another? Consider strategies you might use to support your students in identifying and describing the similarities and differences between representations.
Example 6: *A vertical tangent is drawn on the unit circle. A point (P) moves in an anti-clockwise direction around the circumference of the unit circle.* *The three images below this example show different heights marked on the tangent as P moves through 30° (A), 45° (B) and 60°(C).* *a) What do you notice? What do you wonder?* *A point (D) is drawn on the tangent when the angle is 80°.* *b) Describe the position of D.* *c) What will happen when the angle is 90°?*	*Examples 6* and *7* are a continuation of the thinking in *Example 5*, focusing on the tangent function rather than sine and cosine. Offer students some time to consider the unit circle and graphical **representations**, ensuring they understand the relevant measurements. They need to appreciate how the angle at the centre of the unit circle will change the height at which its line meets the tangent, which is challenging to conceptualise. The prompt in part a is open so that discussions can be led by students. They may compare the heights of A, B and C with the equivalent values in the sine and cosine functions, and notice that, unlike in the other two trigonometric functions, the values of tan(P) extend beyond 1. Part b asks them to consider how this shape will continue, while part c invites them to consider the vertical line at 90, which will never meet the tangent. The **language** of asymptote is not used here so that the focus is on the shape of the function, rather than the technicalities of new terminology. Teachers will need to plan when to introduce this term. How often do you use the prompt to ‘wonder’ in the mathematics classroom? The trigonometric graphs offer a particularly rich context for students to imagine the possibilities. Students will be experiencing something entirely new (a cyclic function) but in the context of something entirely familiar (the

trigonometric ratios). Think about how you can capture students' 'wonderings' and compare them with the reality.



Example 7:

The graph below shows how the height of the point on the tangent changes as P travels one turn anticlockwise around the unit circle. Points A, B and C are marked on both the graph and the circle.

a) *Find another point on the graph with the same value as A. Label this as D on both the graph and the tangent.*

b) *Mark the equivalent position of point P on the unit circle.*

c) *Choose a position for point P in quadrant 2.*

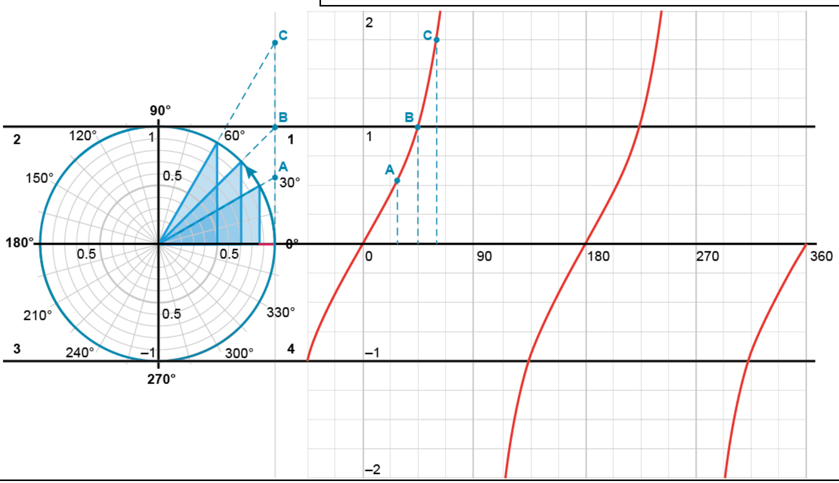
d) *Mark the equivalent position (labelled E) on both the tangent and the graph.*

e) *Find another point with the same value on both the graph and tangent. Label these points as F.*

f) *Mark the equivalent position of point P on the unit circle.*

Example 7 offers an opportunity for **deepening** understanding of the structure of the tangent function. The task is intentionally student-led, so they choose the points to work with. This means that there is opportunity to compare answers and discuss similarities, drawing attention to the properties of the tangent function and the structures that are inherent within it. Students should notice that there are two points with the same value within one rotation of point P. They should also connect point P's position with the height on the tangent. In quadrants 2 and 3, this involves drawing a ray that goes through the origin and across the opposite quadrant before meeting the tangent. Students might find this challenging to visualise and need more prompting.

A vertical tangent has been used here as the most straightforward representation for a series of static images. The tangent function can also be modelled using a series of different tangents and measuring the distance from the point on the circumference to the x-axis. With your department, use dynamic graphing software to explore other ways to model the tangent function. Which approach do you find most accessible? Do your colleagues favour the same one?



Understand that, for the graph of any trigonometric function, there are multiple values of x for a particular value of y

Example 8:

Below are two images, of a unit circle and a graph.

A point travels anticlockwise around the circumference of this circle. The graph shows the height of the point as it travels.

a) After the point has turned through 60°, what is its height?

b) After the point has turned through 190°, what is its height?

c) The height of the point is −0.5. What angle has it turned through?

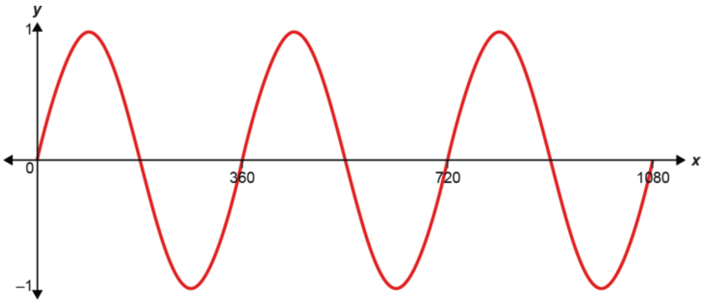
d) How confident are you in your answers for parts a to c? Why?

Ravi says, 'I think it made nearly four complete rotations before it stopped with a height of −0.5.'

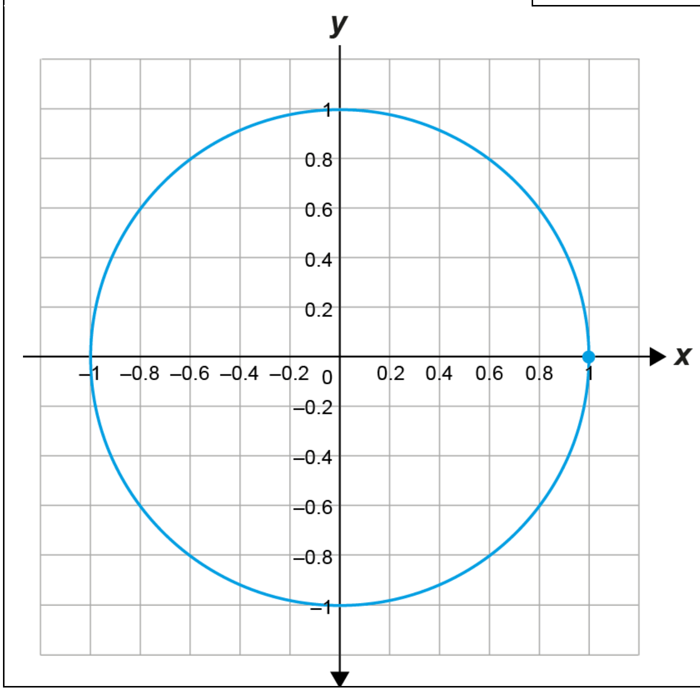
e) What angle do you think the point might have turned through?

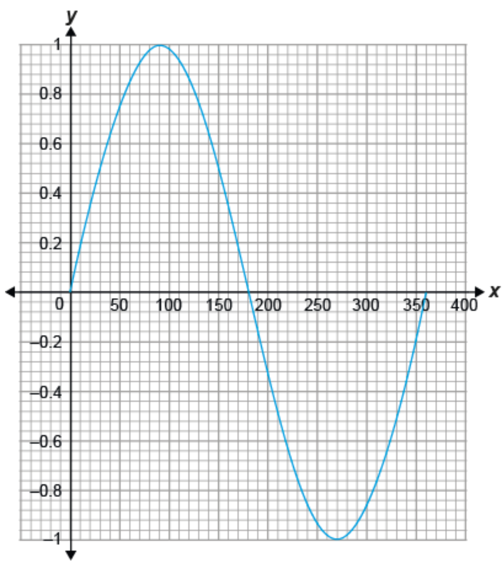
Example 8 draws students' attention to the property of functions that, for any given value of $x, f(x)$ can have just one value. This graphical **representation** of the function makes clear that one angle, as imagined on a unit circle, corresponds to exactly one height. However, as parts b and c make clear, the reverse is not true. The same height can be achieved by turning different angles.

The decision to only show the graph for one complete turn, but to ask about multiple turns in part c, is intended to **deepen** students' understanding of the cyclic nature of this function:



This provides a context in which students need to connect a (perhaps mental) image of the unit circle with the sine curve and make appropriate connections using their knowledge of this function to reach an answer.





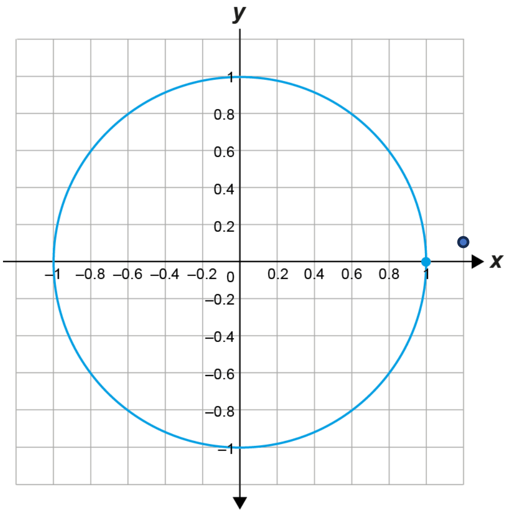
Example 9: *a) For each property in the first five rows in the table below, tick which of the trigonometric functions it applies to.* *b) For each blank row, write a property that applies to the selected trigonometric functions*	*Example 9* is designed for **deepening** students' understanding of the three trigonometric functions. By considering properties that are shared or not shared, students can build a more secure sense of what is an essential feature of each function. Part b offers them an opportunity to notice their own properties, and time could be spent comparing their answers. There will be multiple valid responses for each row, and collating a set of properties that has been generated by the whole class could help students to define each function more clearly. The **language** in this example could be extended to include more terms specific to trigonometric functions, such as 'amplitude' (to describe the height of the function in one revolution of the unit circle) or 'period' (to describe the size of the repeating aspect of the function). To ensure that this example is as accessible as possible, these terms are not introduced here. Teachers can make alterations to reflect the vocabulary they have used with students.

	$Sin(x)$	$Cos(x)$	$Tan(x)$
Is symmetrical between 0° and 360°			
Is positive between 0° and 90°			
Is positive between 90° and 180°			
Is continuous			
Intersects the x-axis at 180°			
			✓
		✓	
	✓		
	✓	✓	
	✓		✓
	✓	✓	✓

Identify key points on the graphs of trigonometric functions and relate them to the unit circle.

Example 10:

Kyle is thinking about a point spinning around the circumference of a unit circle. It starts at (1,0).



a) *What angle will the point have turned when it is at its highest point? What about the lowest point?*

b) *What angle could the point have turned if it has a height of 0.5?*

Kyle sketches the graph of the height of the point above the horizontal axis like the graph shown to the right. He wonders what would happen if he changed some of the conditions.

c) *How does the graph change, and what stays the same, when:*

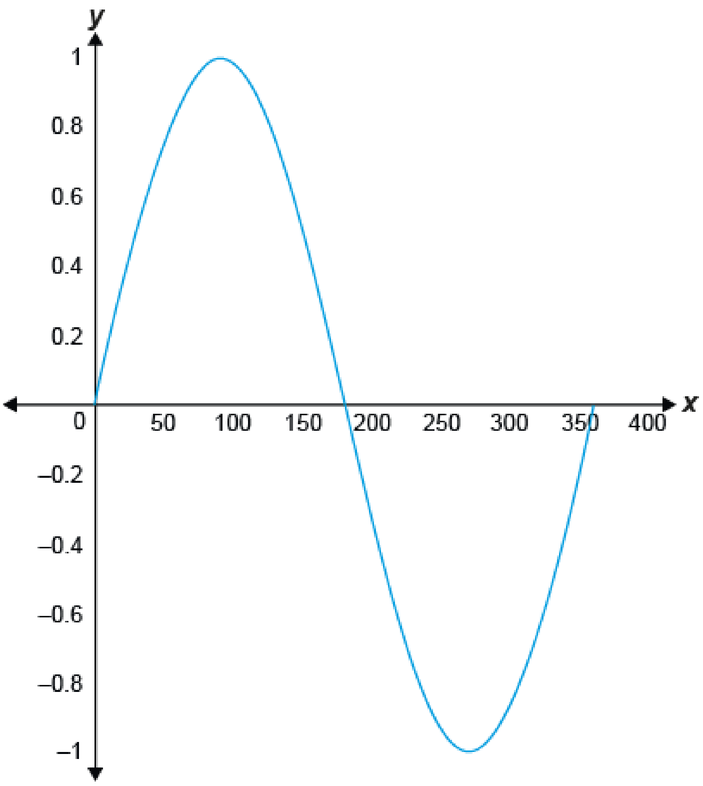
(i) *The circle changes from being a unit circle to having a radius of 2?*

(ii) *The way that the height is measured changes so height above the very bottom of the unit circle is used, rather than height above the horizontal axis?*

(iii) *The speed at which the point travels around the unit circle is halved so it takes double the time to make one full revolution?*

Example 10 uses the context of transformations of the sine curve to consider particular key points on that curve. The intention here is not to use the rules of transformations of graphs, rather to think about the impact that the **variation** in the question will have on some values, and to reason about the way that will affect the shape and size of particular features of the graph.

In doing so, students will **deepen** their understanding of the meaning of these key instances in the different representations and connect those meanings to the function as a whole. Note that part c (iii) does not impact on the shape of the graph since speed is not measured in either axis. Students should be able to reason with this, but a misconception that speed will be impacted on the horizontal axis may be unpacked should it arise.



Collaborative planning

Although they may provoke thought if read and worked on individually, the materials are best worked on with others as part of a collaborative professional development activity based around planning lessons and sequences of lessons.

If being used in this way, is important to stress that they are not intended as a lesson-by-lesson scheme of work. In particular, there is no suggestion that each key idea represents a lesson. Rather, the fine-grained distinctions offered in the key ideas are intended to help you think about the learning journey, irrespective of the number of lessons taught. Not all key ideas are of equal weight. The amount of classroom time required for them to be mastered will vary. Each step is a noteworthy contribution to the statement of knowledge, skills and understanding with which it is associated.

Some of the key ideas have been extensively exemplified in the guidance documents. These exemplifications are provided so that you can use them directly in your own teaching but also so that you can critique, modify and add to them as part of any collaborative planning that you do as a department. The exemplification is intended to be a starting point to catalyse further thought rather than a finished 'product'.

A number of different scenarios are possible when using the materials. You could:

- Consider a collection of key ideas within a core concept and how the teaching of these translates into lessons. Discuss what range of examples you will want to include within each lesson to ensure that enough attention is paid to each step, but also that the connections between them and the overall concepts binding them are not lost.
- Choose a topic you are going to teach and discuss with colleagues the suggested examples and guidance. Then plan a lesson or sequence of lessons together.
- Look at a section of your scheme of work that you wish to develop and use the materials to help you to re-draft it.
- Try some of the examples together in a departmental meeting. Discuss the guidance and use the PD prompts where they are given to support your own professional development.
- Take a key idea that is not exemplified and plan your own examples and guidance using the template available at Resources for teachers using the mastery materials | NCETM.

Remember, the intention of these PD materials is to provoke thought and raise questions rather than to offer a set of instructions.

Solutions

Solutions for all the examples from *Theme 9 Sequences, functions and graphs* can be found here:

https://www.ncetm.org.uk/media/23eejt3r/ncetm_ks4_cc_9_solutions.pdf